

HEAT CONTENT ASYMPTOTICS WITH SINGULAR INITIAL TEMPERATURE DISTRIBUTIONS

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ABSTRACT. We study the heat content asymptotics with either Dirichlet or Robin boundary conditions where the initial temperature exhibits radial blowup near the boundary. We show that there is a complete small-time asymptotic expansion and give explicit geometrical formulas for the first few terms in the expansion.

1. INTRODUCTION

1.1. The heat content. Let M be a compact Riemannian manifold of dimension m with smooth boundary ∂M , let dx and dy be the Riemannian measures on M and on ∂M , respectively, and let D be an operator of Laplace type on a smooth vector bundle V over M . To impose suitable boundary conditions, we define the *Dirichlet boundary operator* $B_D\phi := \phi|_{\partial M}$. The operator D defines a natural connection ∇ as we shall discuss presently in Lemma 1.1. Let S be an auxiliary endomorphism of the vector bundle $V|_{\partial M}$. Let

$$B_R\phi := (\phi_{;m} + S\phi)|_{\partial M}$$

be the *Robin boundary operator* where $\phi_{;m}$ denotes the covariant derivative of ϕ with respect to the inward unit normal vector field. Let B be either the Dirichlet or the Robin boundary operator; the associated boundary conditions are defined by setting $B\phi = 0$. It is well known that the heat equation

$$(\partial_t + D)u_B(x; t) = 0, \quad Bu(\cdot; t) = 0, \quad \lim_{t \downarrow 0} u_B(\cdot; t) = \phi(\cdot),$$

has a unique classical solution for a wide class of initial temperature distributions ϕ . We set $u = e^{-tD_B}\phi$ where D_B is the associated realization of D . The operator e^{-tD_B} has a kernel $p_B(x, \tilde{x}; t)$ which is smooth in $(x, \tilde{x}; t)$ such that

$$u_B(x; t) = \int_M p_B(x, \tilde{x}; t) \phi(\tilde{x}) d\tilde{x}.$$

If, for example, $D = \Delta := \delta d$ is the scalar Laplacian, then one may take a complete spectral resolution $\{\lambda_i, \phi_i\}$ of Δ_B and express

$$p_B(x, \tilde{x}; t) = \sum_i e^{-t\lambda_i} \phi_i(x) \bar{\phi}_i(\tilde{x}).$$

Let $\langle \cdot, \cdot \rangle$ denote the natural pairing between V and the dual bundle \tilde{V} . The *specific heat* ρ of the manifold is a smooth section of \tilde{V} and the *heat content* β is given by:

$$\beta(\phi, \rho, D, B)(t) := \int_M \langle u_B(x; t), \rho(x) \rangle dx = \int_M \int_M \langle p_B(x, \tilde{x}; t) \phi(\tilde{x}), \rho(x) \rangle d\tilde{x} dx.$$

Although in most practical applications it is customary to take V and \tilde{V} to be the trivial line bundle and $D = \Delta$, it is necessary to work in this greater generality as

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we shall see presently in Section 3. If ∂M is empty, we shall omit the boundary condition B from the notation as it plays no role. To simplify the notation, we shall write D_B , p_B , $\beta(\cdot, \cdot, D, B)$, and u_B for the most part except where it is useful to emphasize which boundary condition appears.

1.2. Geometric preliminaries. The following formalism will enable us to work in a tensorial and coordinate free fashion. We adopt the *Einstein convention* and sum over repeated indices. Choose a system of local coordinates $x = (x_1, \dots, x_m)$ for M and choose a local trivialization of V . Let $g_{\mu\nu} := g(\partial_{x_\mu}, \partial_{x_\nu})$ and let $g^{\mu\nu}$ be the inverse matrix. As D is of Laplace type, there are matrices A_1^ν and A_0 so that:

$$(1.a) \quad D = -\{g^{\mu\nu} \text{Id} \partial_{x_\mu} \partial_{x_\nu} + A_1^\nu \partial_{x_\nu} + A_0\}.$$

If ∇ is a connection on V , we use ∇ and the Levi-Civita connection to covariantly differentiate tensors of all types and let ‘;’ denote multiple covariant differentiation. We let $\phi_{;\mu\nu}$ be the components of $\nabla^2 \phi$. If E is an auxiliary endomorphism of V , we define the associated *modified Bochner Laplacian*:

$$D(g, \nabla, E)\phi := -g^{\mu\nu} \phi_{;\nu\mu} - E\phi.$$

Let $\Gamma_{\mu\nu\sigma}$ and $\Gamma_{\mu\nu}^\sigma$ be the Christoffel symbols. We then have [9]:

Lemma 1.1. *If D is an operator of Laplace type, then there exists a unique connection ∇ on V and a unique endomorphism E on V so that $D = D(g, \nabla, E)$. The connection 1-form ω of ∇ and the endomorphism E are given by:*

$$(1) \quad \omega_\mu = \frac{1}{2}(g_{\mu\nu} A_1^\nu + g^{\sigma\varepsilon} \Gamma_{\sigma\varepsilon\mu} \text{Id}).$$

$$(2) \quad E = A_0 - g^{\mu\nu} (\partial_{x_\nu} \omega_\mu + \omega_\mu \omega_\nu - \omega_\sigma \Gamma_{\mu\nu}^\sigma).$$

We use the dual connection to covariantly differentiate the specific heat ρ ; note that the connection 1 form $\tilde{\omega}_\nu$ for $\tilde{\nabla}$ is the dual of $-\omega_\nu$. Thus

$$(1.b) \quad \tilde{\nabla}_{\partial_{x_\mu}} = \partial_{x_\mu} - \frac{1}{2}(g_{\mu\nu} \tilde{A}_1^\nu + g^{\sigma\varepsilon} \Gamma_{\sigma\varepsilon\mu} \text{id}) \quad \text{and} \quad \tilde{D}\rho = -(g^{\mu\nu} \rho_{;\mu\nu} + \tilde{E}\rho).$$

Near the boundary, choose an orthonormal frame $\{e_1, \dots, e_m\}$ for the tangent bundle so that e_m is the inward unit geodesic normal; let indices a, b range from 1 to $m-1$ and index the induced orthonormal frame $\{e_1, \dots, e_{m-1}\}$ for the tangent bundle of the boundary. We let ‘:’ denote the components of tangential covariant differentiation defined by ∇ and the Levi-Civita connection of the boundary. Let $L_{ab} := g(\nabla_{e_a} e_b, e_m) = \Gamma_{abm}$ be the components of the second fundamental form. The difference between ‘;’ and ‘:’ is then measured by L . For example, the following relation will prove useful subsequently:

$$(1.c) \quad \begin{aligned} D\phi &= -(\phi_{;aa} + \phi_{;mm} - L_{aa}\phi_{;m} + E\phi), \\ \tilde{D}\rho &= -(\rho_{;aa} + \rho_{;mm} - L_{aa}\rho_{;m} + \tilde{E}\rho). \end{aligned}$$

Let Ric denote the Ricci tensor. Let \tilde{B}_R be the dual Robin boundary operator; it is defined by the dual connection $\tilde{\nabla}$ and dual endomorphism \tilde{S} .

1.3. Heat content asymptotics in the smooth setting. One can use the calculus of pseudo-differential operators developed in [10, 12, 13] to show that:

Theorem 1.2. *Let $\phi \in C^\infty(V)$ and let $\rho \in C^\infty(\tilde{V})$. There is a complete asymptotic expansion as $t \downarrow 0$ of the form:*

$$\beta(\phi, \rho, D, B)(t) \sim \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \int_M \langle \phi, \tilde{D}^n \rho \rangle dx + \sum_{k=0}^{\infty} t^{(1+k)/2} \int_{\partial M} \beta_k^{\partial M}(\phi, \rho, D, B) dy.$$

The *heat content coefficients* $\beta_k^{\partial M}$ are locally computable and are given by local geometric invariants.

Theorem 1.3. *Let $\phi \in C^\infty(V)$ and let $\rho \in C^\infty(\tilde{V})$.*

- (1) *With Dirichlet boundary conditions, one has that:*

- (a) $\int_{\partial M} \beta_0^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy = -\frac{2}{\sqrt{\pi}} \int_{\partial M} \langle \phi, \rho \rangle dy.$
- (b) $\int_{\partial M} \beta_1^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy = -\int_{\partial M} \{ \langle \phi;_m, \rho \rangle - \frac{1}{2} \langle L_{aa} \phi, \rho \rangle \} dy.$
- (c) $\int_{\partial M} \beta_2^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy = -\frac{2}{\sqrt{\pi}} \int_{\partial M} \{ \langle \frac{2}{3} \phi;_{mm}, \rho \rangle + \frac{2}{3} \langle \phi, \rho;_{mm} \rangle - \frac{2}{3} L_{aa} \langle \phi, \rho \rangle;_m + \langle \phi E, \rho \rangle - \langle \phi;_a, \rho;_a \rangle + \langle (\frac{1}{12} L_{aa} L_{bb} - \frac{1}{6} L_{ab} L_{ab} - \frac{1}{6} \text{Ric}_{mm}) \phi, \rho \rangle \} dy.$
- (2) *With Robin boundary conditions, one has that:*
 - (a) $\int_{\partial M} \beta_0^{\partial M}(\phi, \rho, D, B_{\mathcal{R}}) dy = 0.$
 - (b) $\int_{\partial M} \beta_1^{\partial M}(\phi, \rho, D, B_{\mathcal{R}}) dy = \int_{\partial M} \langle \phi, \tilde{B}_{\mathcal{R}} \rho \rangle dy.$
 - (c) $\int_{\partial M} \beta_2^{\partial M}(\phi, \rho, D, B_{\mathcal{R}}) dy = \frac{4}{3\sqrt{\pi}} \int_{\partial M} \langle B_{\mathcal{R}} \phi, \tilde{B}_{\mathcal{R}} \rho \rangle dy.$

With Dirichlet boundary conditions, $\beta_3^{\partial M}$ is known and partial information concerning $\beta_4^{\partial M}$ is available. With Neumann boundary conditions, β_3 , β_4 , and β_5 are known. We refer to [9] for further details; our indexing convention here is slightly different from that employed in [9].

1.4. Singular initial temperatures. We refer to [2] for earlier work in the singular setting as the results of that paper provide the motivation and the starting point for this present work. We begin by using the geodesic flow defined by the unit inward normal vector field to define a diffeomorphism for some $\varepsilon > 0$ between the collar $\mathcal{C}_\varepsilon := \partial M \times [0, \varepsilon]$ and a neighborhood of the boundary in M which identifies $\partial M \times \{0\}$ with ∂M ; the curves $r \rightarrow (y_0, r)$ for $r \in [0, \varepsilon]$ are then unit speed geodesics perpendicular to the boundary and r is the geodesic distance to the boundary. We fix a smooth cutoff function $\chi = \chi(r)$ on \mathcal{C}_ε so that $\chi = 1$ near $r = 0$ and so that $\chi = 0$ near $r = \varepsilon$.

Let ∇ be a connection on a bundle W over \mathcal{C}_ε . Let $\psi \in C^\infty(W|_{\partial M})$. We use parallel translation along the normal geodesic rays to extend ψ to a section of W over \mathcal{C}_ε . We shall denote this extension by $\psi(y)$ to emphasize the fact that $\nabla_{\partial_r} \psi = 0$ on \mathcal{C}_ε . If $W = V$, we use the connection ∇ defined by D ; if $W = \tilde{V}$, we use the dual connection $\tilde{\nabla}$ defined by \tilde{D} . We refer to [9] for further details.

Fix $\alpha \in \mathbb{C}$. Let ϕ be a smooth section to V on the interior of M such that $\phi r^\alpha \in C^\infty(\mathcal{C}_\varepsilon)$; the parameter α controls the growth (if $\text{Re}(\alpha) > 0$) or decay (if $\text{Re}(\alpha) < 0$) of ϕ near the boundary assuming that ϕr^α does not vanish identically on the boundary. We assume the specific heat is smooth so $\rho \in C^\infty(\tilde{V})$. We may then expand ϕ and ρ on \mathcal{C}_ε in the form:

$$(1.d) \quad \phi(y, r) \sim \sum_{i=0}^{\infty} \phi_i(y) r^{i-\alpha} \quad \text{and} \quad \rho(y, r) \sim \sum_{i=0}^{\infty} \rho_i(y) r^i \quad \text{as } r \downarrow 0.$$

The coefficients ϕ_i and ρ_i are then uniquely specified by the requirement that $\nabla_{\partial_r} \phi_i = 0$ and $\tilde{\nabla}_{\partial_r} \rho_i = 0$. If $D = \Delta$, then the associated connection is flat and, if $\alpha = 0$, the expansion of Equation (1.d) is just the usual Taylor series expansion of the functions ϕ and ρ . In particular,

$$\begin{aligned} \phi_0 &= (r^\alpha \phi)|_{\partial M}, & \phi_1 &= \{\nabla_{\partial_r} (r^\alpha \phi)\}|_{\partial M}, & \phi_2 &= \frac{1}{2} \{(\nabla_{\partial_r})^2 (r^\alpha \phi)\}|_{\partial M} \\ \rho_0 &= \rho|_{\partial M}, & \rho_1 &= \{\tilde{\nabla}_{\partial_r} \rho\}|_{\partial M}, & \rho_2 &= \frac{1}{2} \{(\tilde{\nabla}_{\partial_r})^2 \rho\}|_{\partial M}. \end{aligned}$$

Fix $t > 0$. Let $x \in M$ and let $\tilde{x} = (\tilde{y}, r) \in \mathcal{C}_\varepsilon$. Suppose first that $B = B_{\mathcal{D}}$ defines Dirichlet boundary conditions. Then $p_B(x, (\tilde{y}, \tilde{r}), t)|_{\tilde{r}=0} = 0$. Since p_B is smooth for $t > 0$ and \mathcal{C}_ε is compact, we may use the Taylor series expansion of p_B to derive the estimate:

$$|p_B(x, (\tilde{r}, \tilde{y}); t)| \leq C(t) \tilde{r} \quad \text{on } \mathcal{C}_\varepsilon.$$

If $\text{Re}(\alpha) < 2$, then the integral $u_B(x; t) = \int_M p_B(x, \tilde{x}; t) \phi(\tilde{x}) d\tilde{x}$ is convergent and bounded in x . Consequently the heat content

$$\beta(\phi, \rho, D, B)(t) := \int_M \langle u_B(x; t), \rho(x) \rangle dx$$

is well defined for $t > 0$. If $\operatorname{Re}(\alpha) < 1$, then $\phi \in L^1$ and the initial heat content $\beta(\phi, \rho, D, B)(0) = \int_M \langle \phi, \rho \rangle dx$ is finite. If, however, $1 \leq \operatorname{Re}(\alpha)$, then this integral may be divergent and the initial heat content can be infinite. Still, sufficient cooling near the boundary takes place for u_B to be in L^1 for any $t > 0$. A similar phenomenon occurs in the setting of non-compact Riemannian manifolds with infinite volume and with regular boundary and initial temperature $\phi = 1$ [1, 4].

As this cooling phenomenon does not occur with Robin boundary conditions $B = B_{\mathcal{R}}$, we shall always assume $\operatorname{Re}(\alpha) < 1$ in this instance.

It is important to observe that although we are primarily interested in positive real α , it is necessary to consider complex values of α to justify some analytic continuation arguments. It is also necessary to permit $\operatorname{Re}(\alpha) < 0$ to justify some computations in Sections 3 and 4; these values are of interest in their own right since ϕ is not smooth if α is not an integer.

If $1 \leq \operatorname{Re}(\alpha) < 2$, we must regularize the integral $\int_M \langle \phi, \rho \rangle dx$ since the integral may be divergent. The Riemannian measure is not in general product near the boundary. Since, however, $dx = dydr$ on the boundary of M , we may decompose

$$\langle \phi, \rho \rangle dx = \langle \phi_0, \rho_0 \rangle r^{-\alpha} dydr + O(r^{1-\alpha}).$$

For $\operatorname{Re}(\alpha) < 2$, define:

$$(1.e) \quad \begin{aligned} \mathcal{I}_{\text{Reg}}(\phi, \rho) &:= \int_{M-\mathcal{C}_\varepsilon} \langle \phi, \rho \rangle dx + \int_{\mathcal{C}_\varepsilon} \{ \langle \phi, \rho \rangle dx - \langle \phi_0, \rho_0 \rangle r^{-\alpha} dydr \} \\ &+ \int_{\partial M} \langle \phi_0, \rho_0 \rangle dy \times \begin{cases} \frac{\varepsilon^{1-\alpha}}{1-\alpha} & \text{if } \alpha \neq 1, \\ \ln(\varepsilon) & \text{if } \alpha = 1. \end{cases} \end{aligned}$$

This is clearly independent of ε and agrees with $\int_M \langle \phi, \rho \rangle$ if $\operatorname{Re}(\alpha) < 1$. Briefly, the regularization $\mathcal{I}_{\text{Reg}}(\phi, \rho)$ is a meromorphic function of α with a simple pole at $\alpha = 1$. When $\alpha = 1$, then $\mathcal{I}_{\text{Reg}}(\phi, \rho)$ is defined as the constant term in the Laurent expansion at $\alpha = 1$, thus dropping the pole.

The following is the main analytic result of this paper:

Theorem 1.4. *If $B = B_{\mathcal{D}}$, assume $\operatorname{Re}(\alpha) < 2$; if $B = B_{\mathcal{R}}$, assume $\operatorname{Re}(\alpha) < 1$.*

(1) *If $\alpha \neq 1$, then there exists a full asymptotic expansion as $t \downarrow 0$ of the form:*

$$\begin{aligned} \beta(\phi, \rho, D, B)(t) &\sim \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \mathcal{I}_{\text{Reg}}(\phi, \tilde{D}^n \rho) \\ &+ \sum_{k=0}^{\infty} t^{(1+k-\alpha)/2} \int_{\partial M} \beta_{k,\alpha}^{\partial M}(\phi, \rho, D, B) dy. \end{aligned}$$

(2) *If $\alpha = 1$ (and hence $B = B_{\mathcal{D}}$), then there exists a full asymptotic expansion as $t \downarrow 0$ of the form:*

$$\begin{aligned} \beta(\phi, \rho, D, B)(t) &\sim \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \mathcal{I}_{\text{Reg}}(\phi, \tilde{D}^n \rho) \\ &+ \sum_{k=0}^{\infty} t^{k/2} \int_{\partial M} \{ \beta_{k,1}^{\partial M}(\phi, \rho, D, B) + \ln(t) \check{\beta}_k^{\partial M}(\phi, \rho, D, B) \} dy. \end{aligned}$$

(3) *There are natural tangential bilinear differential operators $\beta_{k,\alpha,i,j}^{\partial M}$, which are holomorphic for $\alpha \neq 1$, so that*

$$\beta_{k,\alpha}^{\partial M}(\phi, \rho, D, B) = \sum_{i+j \leq k} \beta_{k,\alpha,i,j}^{\partial M}(\phi_i, \rho_j, D, B).$$

$$(4) \quad \int_{\partial M} \check{\beta}_k^{\partial M}(\phi, \rho, D, B) dy = \begin{cases} -\frac{1}{2} \frac{(-1)^n}{n!} \int_{\partial M} \langle \phi_0, (\tilde{D}^n \rho)_0 \rangle dy & \text{if } k = 2n, \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

We note that $\tilde{B}_R \rho = \rho_1 + \tilde{S} \rho_0$. Theorem 1.3 generalizes to this setting to become:

Theorem 1.5. Set $c_\alpha := 2^{1-\alpha} \Gamma\left(\frac{2-\alpha}{2}\right) \frac{1}{\sqrt{\pi}(\alpha-1)}$.

- (1) If $\alpha \neq 1$, if $\text{Re}(\alpha) < 2$, and if $B = B_{\mathcal{D}}$, then:
 - (a) $\int_{\partial M} \beta_{0,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy = c_\alpha \int_{\partial M} \langle \phi_0, \rho_0 \rangle dy.$
 - (b) $\int_{\partial M} \beta_{1,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy = c_{\alpha-1} \int_{\partial M} \langle \phi_1 - \frac{1}{2} L_{aa} \phi_0, \rho_0 \rangle dy.$
 - (c) $\int_{\partial M} \beta_{2,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy = c_{\alpha-2} \int_{\partial M} \{ \langle \phi_2, \rho_0 \rangle - \frac{1}{2} \langle L_{aa} \phi_1, \rho_0 \rangle$
 $- \frac{\alpha-3}{2(\alpha-1)(\alpha-2)} \langle E \phi_0, \rho_0 \rangle + \frac{2}{(\alpha-1)(\alpha-2)} \langle \phi_0, \rho_2 \rangle - \frac{1}{(\alpha-1)(\alpha-2)} \langle L_{aa} \phi_0, \rho_1 \rangle$
 $+ \frac{\alpha-3}{2(\alpha-1)(\alpha-2)} \langle \phi_{0:a}, \rho_{0:a} \rangle - \frac{\alpha-1}{4(\alpha-2)} \langle \text{Ric}_{mm} \phi_0, \rho_0 \rangle$
 $+ \frac{\alpha-1}{8(\alpha-2)} \langle L_{aa} L_{bb} \phi_0, \rho_0 \rangle - \frac{\alpha-1}{4(\alpha-2)} \langle L_{ab} L_{ab} \phi_0, \rho_0 \rangle \} dy.$
- (2) Let γ be Euler's constant. If $\alpha = 1$ and if $B = B_{\mathcal{D}}$, then

$$\begin{aligned} \beta(\phi, \rho, D, B_{\mathcal{D}})(t) &\sim \mathcal{I}_{\text{Reg}}(\phi, \rho) - \frac{1}{2} \ln(t) \int_{\partial M} \langle \phi_0, \rho_0 \rangle dy + \int_{\partial M} \frac{\gamma}{2} \langle \phi_0, \rho_0 \rangle dy \\ &+ t^{1/2} \int_{\partial M} \{ -\frac{2}{\sqrt{\pi}} \langle \phi_1, \rho_0 \rangle + \frac{1}{\sqrt{\pi}} \langle L_{aa} \phi_0, \rho_0 \rangle \} dy \\ &+ t \{ -\mathcal{I}_{\text{Reg}}(\phi, \tilde{D} \rho) + \frac{1}{2} \ln(t) \int_{\partial M} \langle \phi_0, (\tilde{D} \rho)_0 \rangle dy \} \\ &+ t \int_{\partial M} \{ \frac{\gamma}{2} \langle \phi_0, -(\tilde{D} \rho)_0 \rangle - \langle \phi_2, \rho_0 \rangle + \frac{1}{2} \langle L_{aa} \phi_1, \rho_0 \rangle + \langle \phi_0, \rho_2 \rangle \\ &- \frac{1}{2} \langle L_{aa} \phi_0, \rho_1 \rangle \} dy + O(t^{3/2}). \end{aligned}$$
- (3) If $\alpha \neq 0$, if $(\text{Re})(\alpha) < 1$, and if $B = B_{\mathcal{R}}$, then:
 - (a) $\int_{\partial M} \beta_{0,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}}) dy = 0.$
 - (b) $\int_{\partial M} \beta_{1,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}}) dy = \frac{2\alpha}{2-\alpha} c_{\alpha+1} \int_{\partial M} \langle \phi_0, \tilde{B}_R \rho \rangle dy.$
 - (c) $\int_{\partial M} \beta_{2,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}}) dy$
 $= \frac{-2}{3-\alpha} c_\alpha \int_{\partial M} \langle (1-\alpha) \phi_1 + S \phi_0 - \frac{\alpha}{2} L_{aa} \phi_0, \tilde{B}_R \rho \rangle dy.$

Remark 1.6. We note that setting $\alpha = 0$ in Assertion (1) of Theorem 1.5 yields Assertion (1) of Theorem 1.3 and that taking the limit as $\alpha \rightarrow 0$ in Assertion (3) of Theorem 1.5 yields Assertion (2) of Theorem 1.3.

To avoid subscripts on subscripts, we shall for the most part simply refer to u_B , D_B , and p_B when no danger of confusion is likely to ensue; however, we shall use the notation $D_{B_{\mathcal{D}}}$ and $D_{B_{\mathcal{R}}}$ in Section 4 when we must deal with two different boundary conditions.

1.5. Outline of the paper. In Section 2, we use the calculus of pseudo-differential operators to establish Theorem 1.4 (1-3); we postpone the proof of (4) as it will follow as a scholium to the proof of Lemma 3.1; it can also be deduced directly from Lemma 2.6. We shall restrict to Dirichlet boundary conditions as the analysis is similar for Robin boundary conditions. In Section 3, we apply invariance theory and the functorial method to prove Assertions (1) and (2) of Theorem 1.5; an essential input is the calculation of a single coefficient performed in [2] and a special case computation on the half-line. In Section 4, we establish Assertion (3) of Theorem 1.5. We plan in a subsequent paper [5] to undertake a similar analysis of the heat trace asymptotics with a singular smearing function.

The special case $\alpha = 1$ in Theorem 1.5 can be derived from the case $\alpha \neq 1$ by a straightforward but tedious analytic continuation; the poles at $\alpha = 1$ which arise from the terms in Assertion 1 of Theorem 1.5 are cancelled by those involved in the regularization $\mathcal{I}_{\text{Reg}}(\phi, \tilde{D}^n \rho)$. The derivation of the t^0 and $t^0 \ln(t)$ terms in this way is sketched in Section 2.1; the other terms use Equations (1.b), (1.c), and (3.b) below. We will give an independent derivation of the case $\alpha = 1$ to show how scaling arguments can be applied when logarithmic terms are present.

2. PSEUDO-DIFFERENTIAL OPERATORS

We suppose throughout Section 2 that $\operatorname{Re}(\alpha) < 2$ and that $B = B_{\mathcal{D}}$. We use the identity

$$\int_M \langle e^{-tD_B} \phi, \rho \rangle dx = \int_M \langle \phi, e^{-t\tilde{D}_B} \rho \rangle dx$$

to interchange the roles of ϕ and ρ . This causes no difficulty as

$$\beta(\phi, \rho, D, B_{\mathcal{D}})(t) = \beta(\rho, \phi, \tilde{D}, B_{\mathcal{D}})(t).$$

We shall assume throughout this section that ϕ is smooth and that

$$\rho(y, r) = \rho_0(y) \chi(r) r^{-\alpha} \quad \text{on } \mathcal{C}_\epsilon.$$

The more general case where $\rho \sim (\rho_0(y) + \rho_1(y)r + \dots) \chi(r) r^{-\alpha}$ then follows similarly; if ϕ and ρ vanish to high order on the boundary, the corresponding boundary contributions vanish to high order in t . Frequently in this section, we will let β be a multi-index rather than denoting the total heat content; we shall also let γ both be a multi-index and Euler's constant. We apologize in advance for any confusion this may cause. We let \mathcal{I}_{Reg} be the regularization defined in Equation (1.e) where we interchange the roles of ϕ and ρ .

Our fundamental analytical result is the following:

Theorem 2.1. *Adopt the notation established above. There are differential operators \mathbb{G}_k of order at most k defined on M and tangential differential operators $\mathbb{B}_k(\alpha)$ and \mathbb{L}_k of order at most k defined on ∂M such that in closed subsectors of $\{\mu \in \mathbb{C} : |\arg(\mu)| < \frac{1}{2}\pi\}$ one has:*

- (1) *Let $\alpha \neq 1$ and let $\operatorname{Re}(\alpha) < 2$. Then $\mathbb{B}_0(\alpha) = -\Gamma(1 - \alpha)$, $\mathbb{B}_k(\alpha)$ is holomorphic in α , and*

$$\langle (D_B + \mu^2)^{-1} \phi, \rho \rangle \sim \sum_{k=0}^{\infty} \mu^{-2-k} \mathcal{I}_{\text{Reg}}(\mathbb{G}_k \phi, \rho) + \sum_{k=0}^{\infty} \mu^{\alpha-3-k} \int_{\partial M} \langle \mathbb{B}_k(\alpha) \phi, \rho_0 \rangle dy.$$

- (2) *If $\alpha = 1$, then:*

$$\begin{aligned} \langle (D_B + \mu^2)^{-1} \phi, \rho \rangle &\sim \sum_{k=0}^{\infty} \mu^{-2-k} \left\{ \mathcal{I}_{\text{Reg}}(\mathbb{G}_k \phi, \rho) + \int_{\partial M} \langle \mathbb{B}_k(1) \phi, \rho_0 \rangle dy \right. \\ &\quad \left. + \ln(\mu) \int_{\partial M} \langle \mathbb{L}_k \phi, \rho_0 \rangle dy \right\}. \end{aligned}$$

Here is a brief outline of the proof. In Section 2.1, we study the half-line \mathbb{R}_+^1 . In Section 2.2, we study the half-space \mathbb{R}_+^n . In Section 2.3, we complete the proof by considering the case of manifolds. Theorem 1.4 then follows. The interior integrals are evaluated using previous techniques. Furthermore, for $\alpha \neq 1$, one has

$$\beta_{k,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) = \Gamma\left(\frac{3-\alpha+k}{2}\right)^{-1} \langle \mathbb{B}_k(\alpha) \phi, \rho_0 \rangle.$$

In particular, we may use the duplication formula for the Gamma function to establish Assertion (1a) in Theorem 1.5 by computing:

$$\beta_{0,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) = -\frac{\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{3-\alpha}{2})} \langle \phi_0, \rho_0 \rangle = 2^{1-\alpha} \Gamma\left(\frac{2-\alpha}{2}\right) \frac{1}{\sqrt{\pi}(\alpha-1)} \langle \phi_0, \rho_0 \rangle.$$

2.1. The half line. The case of \mathbb{R}_+^1 gives the basic outline of the proof in the general case. Let η be the Fourier transform variable related to r . The Fourier inversion formula then becomes:

$$\hat{\phi}(\eta) = \int_0^\infty e^{-\sqrt{-1}r\eta} \phi(r) dr \quad \text{and} \quad \phi(r) = \int_{-\infty}^\infty e^{\sqrt{-1}r\eta} \hat{\phi}(\eta) d\eta$$

where we set $\bar{d}\eta := d\eta/2\pi$. For $D = -\partial_r^2$, we may then write the Dirichlet resolvent as a pseudo-differential operator in the form:

$$\begin{aligned} (D_B + \mu^2)^{-1}\phi(r) &= \int_{-\infty}^{\infty} e^{\sqrt{-1}r\eta} c_{-2}(\eta, \mu) \hat{\phi}(\eta) \bar{d}\eta + \int_{-\infty}^{\infty} d_{-2}(r, \eta, \mu) \hat{\phi}(\eta) \bar{d}\eta \\ &=: [\text{Op}(c_{-2}) + \text{Op}'(d_{-2})]\phi(r). \end{aligned}$$

Here $c_{-2} = (\eta^2 + \mu^2)^{-1}$, and $d_{-2}(r, \eta, \mu) = -(\eta^2 + \mu^2)^{-1} e^{-\mu r}$ is the bounded solution of

$$(-\partial_r^2 + \mu^2)d_{-2} = 0 \quad \text{and} \quad d_{-2}|_{r=0} = -c_{-2}(\eta, \mu).$$

The kernel $k(r, s, \mu)$ of $(\Delta_B + \mu^2)^{-1}$ is thus, for $r > 0$ and $s > 0$, given by

$$k(r, s, \mu) = \frac{1}{2\mu} \left[e^{-|r-s|\mu} - e^{-(r+s)\mu} \right].$$

We now consider the heat conduction problem:

$$\phi(r) \equiv 1 \quad \text{and} \quad \rho(r) = \chi(r)r^{-\alpha}, \quad \text{where } \chi \text{ is smooth and}$$

$$\chi(r) = \begin{cases} 1, & 0 < r < 1/3, \\ 0, & 2/3 < r. \end{cases}$$

Suppose that $\text{Re}(\alpha) < 1$. One then has that:

$$\begin{aligned} (2.a) \quad \mu^2 \langle (D_B + \mu^2)^{-1} \phi, \rho \rangle &= \mu^2 \int_0^\infty \chi(r) r^{-\alpha} \int_0^\infty k(r, s, \mu) ds dr \\ &= \int_0^\infty \chi(r) r^{-\alpha} (1 - e^{-r\mu}) dr \\ &= \int_0^1 r^{-\alpha} dr + \int_0^1 (\chi(r) - 1) r^{-\alpha} dr - \int_0^\infty r^{-\alpha} e^{-r\mu} dr \\ &\quad - \int_{\frac{1}{3}}^\infty (\chi(r) - 1) e^{-r\mu} r^{-\alpha} dr \\ &= \left[\frac{1}{(1-\alpha)} + \int_0^1 (\chi(r) - 1) r^{-\alpha} dr \right] - \mu^{\alpha-1} \Gamma(1-\alpha) + O(\mu^{-\infty}). \end{aligned}$$

The term in bracket gives $\mathcal{I}_{\text{Reg}}(\phi, \rho)$ for all $\alpha \neq 1$, hence

$$(2.b) \quad \mu^2 \langle (D_B + \mu^2)^{-1} \phi, \rho \rangle = \mathcal{I}_{\text{Reg}}(\phi, \rho) - \mu^{\alpha-1} \Gamma(1-\alpha) + O(\mu^{-\infty}), \quad \alpha \neq 1$$

As $\alpha \rightarrow 1$ we note that $\Gamma(1-\alpha) = \frac{1}{1-\alpha} - \gamma + \dots$ where γ is Euler's constant. Consequently

$$1/(1-\alpha) - \mu^{\alpha-1} \Gamma(1-\alpha) \rightarrow \ln(\mu) + \gamma.$$

So as $\alpha \rightarrow 1$, Equation (2.a) gives

$$(2.c) \quad \mu^2 \langle (D_B + \mu^2)^{-1} \phi, \rho \rangle = \ln(\mu) + \gamma + \mathcal{I}_{\text{Reg}}(\phi, \rho) + O(\mu^{-\infty}), \quad \alpha = 1.$$

We pass to the heat content by a contour integral

$$\beta(\phi, \rho, D, B_{\mathcal{D}})(t) = \frac{1}{2\pi\sqrt{-1}} \int_C e^{t\lambda} \langle (D_B + \lambda)^{-1} \phi, \rho \rangle d\lambda,$$

where C is a “keyhole contour” consisting of two rays $\{re^{\pm\sqrt{-1}(\pi-\epsilon)}, r \geq R\}$ and a circular arc $\{Re^{\sqrt{-1}\theta}, |\theta| \leq \pi - \epsilon\}$. We then get

$$(2.d) \quad \beta(\phi, \rho, -\partial_r^2, B_D)(t) = \begin{cases} \mathcal{I}_{\text{Reg}}(\phi, \rho) - \frac{\Gamma(1-\alpha)}{\Gamma(\frac{1}{2}(3-\alpha))} + O(t^\infty), & (\alpha \neq 1) \\ \mathcal{I}_{\text{Reg}}(\phi, \rho) - \frac{1}{2} \ln(t) + \frac{1}{2} \gamma + O(t^\infty), & (\alpha = 1) \end{cases}$$

This last formula is valid also for the same functions ϕ and ρ on the interval $[0, 1]$ with Dirichlet conditions at both ends; since $\rho \equiv 0$ near $r = 1$, the boundary correction from $r = 1$ is $O(t^\infty)$. The constant appearing in (2.d) for $\alpha \neq 1$ jibes with the constant c_α defined in Theorem 1.5 by Legendre's duplication formula.

Thus, for this special case, we have established Assertions (1) and (2) of Theorem 1.5.

2.2. The half-space \mathbb{R}_+^m . We use the resolvent construction described in [13]. Let $\xi = (\xi_1, \dots, \xi_{m-1})$ be the Fourier transform variables dual to $y = (y_1, \dots, y_{m-1})$. Note that $g(\partial_{y_i}, \partial_r) = 0$ and $g(\partial_r, \partial_r) = 1$. We set

$$q^2(x, \xi) := g(\xi_a dy^a, \xi_b dy^b) = g^{ab}(x) \xi_a \xi_b.$$

We adopt the notation of Equation (1.a). The symbol of D is given by

$$\begin{aligned} \sigma(D) &= \sigma_2 + \sigma_1 + \sigma_0, & \text{where } \sigma_2 &= \{q^2(x, \xi) + \eta^2\} \text{ id,} \\ \sigma_1 &= -\sqrt{-1}\{A_1^a \xi_a + A_1^m \eta\}, & \text{and } \sigma_0 &= -A_0. \end{aligned}$$

The resolvent parametrix is the sum of an interior part and a boundary correction. Let $y \cdot \xi = y_1 \xi_1 + \dots + y_{m-1} \xi_{m-1}$. The interior part of the parametrix is a finite sum $\text{Op}(c_{-2}) + \dots + \text{Op}(c_{-N})$ where

$$(2.e) \quad [\text{Op}(c)\phi](y, r) = \int \int e^{\sqrt{-1}(y \cdot \xi + r\eta)} c(y, r, \xi, \eta) \hat{\phi}(\xi, \eta) \bar{d}\eta \bar{d}\xi.$$

The leading term in the interior of M is

$$c_{-2}(y, r, \xi, \eta, \mu) = (\sigma_2 + \mu^2)^{-1}$$

and successive terms c_{-3}, \dots are defined by the usual pseudo-differential calculus with parameter μ ; see, for example, Lemma 1.7.2 [8]. In particular, c_j is homogeneous of degree j in the variables (ξ, η, μ) and, for $|\arg(\mu)| \leq \frac{1}{2}\pi - \epsilon$, we have the estimate:

$$(2.f) \quad |\partial_{(y,r)}^\beta \partial_{(\xi,\eta)}^\gamma c_j| \leq \text{const}_{\beta,\gamma,\epsilon,j} (|\xi| + |\eta| + |\mu|)^{j-|\gamma|}.$$

The boundary part of the parametrix is a finite sum of operators

$$\text{Op}'(d_j) = \int_{\mathbb{R}} \int_{\mathbb{R}^{m-1}} e^{\sqrt{-1}y \cdot \xi} d_j(y, r, \xi, \eta, \mu) \hat{\phi}(\xi, \eta) \bar{d}\xi \bar{d}\eta$$

which are chosen so that

- (1) $D\{\text{Op}'(d_{-2}) + \dots + \text{Op}'(d_{-N})\}$ has order $1 - N$,
- (2) $[\text{Op}(c_j) + \text{Op}'(d_j)]\phi(y, 0) = 0$.

To ensure that this second condition is satisfied, we set:

$$(2.g) \quad d_j(y, 0, \xi, \eta, \mu) = -c_j(y, 0, \xi, \eta, \mu).$$

To ensure the first condition is satisfied, we begin by setting

$$[-\partial_r^2 + q^2 + \mu^2]d_{-2} = 0.$$

Then equation (2.g) yields

$$d_{-2}(y, r, \xi, \eta, \mu) = -(q^2 + \eta^2 + \mu^1)^{-1} e^{-r\sqrt{q^2 + \mu^2}}.$$

We may then define d_j inductively by an equation of the form:

$$(-\partial_r^2 + q^2 + \mu^2)d_{j-1} = \sum_{k=2}^{-j} a_{jkl\alpha\beta\gamma}(y) r^\ell \partial_r^\alpha \xi^\beta \partial_y^\gamma d_k,$$

where $k - \ell + u + |\beta| = j + 1$. The coefficients $a_{jkl\alpha\beta\gamma}(y)$ come from the Taylor expansion of the coefficients of D in powers of r , see [13] for details. For some constant C , we have:

$$(2.h) \quad d_j(y, r/t, t\xi, t\eta, t\mu) = t^j d_j(y, r, \xi, \eta, \mu),$$

$$(2.i) \quad r^k \partial_\eta^\alpha \partial_\xi^\beta \partial_r^\ell \partial_y^\gamma d_j = O(|\xi| + |\eta| + |\mu|)^{-2-u} (|\xi| + |\mu|)^{j-k-|\beta|+\ell+2} e^{-Cr(|\xi|+|\mu|)}.$$

Now consider the expansion of $[\text{Op}(c_j)\phi](y, r)$ for $\phi \in \mathcal{S}(\mathbb{R}_+^m)$, i.e. assume that ϕ has an extension in the Schwarz class $\mathcal{S}(\mathbb{R}^m)$. Set

$$\phi^{(\beta)} := \partial_{(y,r)}^\beta \phi(y, r)$$

and

$$c_{j\beta}(y, r, \xi, \eta, \mu) := (-\sqrt{-1})^{|\beta|} \partial_{(\xi, \eta)}^\beta c_j(y, r, \xi, \eta, \mu).$$

Lemma 2.2. *As $\mu \rightarrow \infty$ in closed subsectors of $\{\mu \in \mathbb{C} : |\arg(\mu)| < \frac{1}{2}\pi\}$,*

$$[\text{Op}(c_j)\phi](y, r) \sim \sum_{\beta} \frac{1}{\beta!} [C_{j\beta} + A_{j\beta}] \phi^{(\beta)}(y, r), \quad \text{with}$$

$$C_{j\beta}(y, r, \mu) = c_{j\beta}(y, r, 0, 0, \mu), \quad \text{and}$$

$$A_{j\beta}(y, r, \mu) = - \int_{-\infty}^{-r} \int_{-\infty}^{\infty} e^{-\sqrt{-1}s\zeta} c_{j\beta}(y, r, 0, \zeta, \mu) \bar{d}\zeta ds.$$

Proof. In Equation (2.e), consider first the integral $\bar{d}\xi$. Let $\Phi(y, \eta)$ be the Fourier transform in r . A Taylor expansion of $\Phi(y, \eta)$ in powers of $y - \tilde{y}$ gives

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{\sqrt{-1}y \cdot \xi} c_j(y, r, \xi, \zeta, \mu) \int_{-\infty}^{\infty} e^{-\sqrt{-1}\tilde{y} \cdot \xi} \Phi(\tilde{y}, \zeta) d\tilde{y} \bar{d}\xi \\ &= \sum_{|\gamma| < K} \frac{1}{\gamma!} c_{j\gamma}(y, r, 0, \zeta, \mu) \Phi^{(\gamma)}(y, \zeta) \\ (2.j) \quad &+ \int \int e^{\sqrt{-1}(y-\tilde{y}) \cdot \xi} \sum_{|\gamma|=K} c_{j\gamma}(y, r, \xi, \zeta, \mu) R_{\gamma}(y, r, \tilde{y}, \zeta) \bar{d}\xi d\tilde{y}. \end{aligned}$$

In this expression, $\gamma = (\gamma_1, \dots, \gamma_{m-1}, 0)$ is a multi-index. The change in the order of integration is clearly justified if $j < 1 - m$. For other j , we may insert a factor of $(1 + |\xi|^2)^w$ and continue analytically to $w = 0$ from $2\text{Re}(w) + j < 1 - m$.

We now multiply Equation (2.j) by $e^{\sqrt{-1}r\zeta}$ and integrate $\bar{d}\zeta$. The remainder integral is a harmless $O(|\mu|^{j-k+m+2})$ since R_j is bounded and for all γ' and for $|\gamma| = K$, we have

$$(y - \tilde{y})^{\gamma'} \int e^{\sqrt{-1}(y-\tilde{y}) \cdot \xi} c_{j\gamma}(y, r, \xi, \zeta, \mu) \bar{d}\xi = O(|\zeta| + |\mu|)^{j-k+m}.$$

From the terms with $|\gamma| < K$, we have

$$\begin{aligned} & \frac{1}{\gamma!} \int_{-\infty}^{\infty} e^{\sqrt{-1}r\zeta} c_{j\gamma}(y, r, 0, \zeta, \mu) \Phi^{(\gamma)}(y, \zeta) \bar{d}\zeta \\ &= \frac{1}{\gamma!} \int_0^{\infty} \int_{-\infty}^{\infty} e^{\sqrt{-1}(r-s)\zeta} c_{j\gamma}(y, r, 0, \zeta, \mu) \phi^{(\gamma)}(y, s) \bar{d}\zeta ds. \end{aligned}$$

A Taylor expansion of $\phi^{(\gamma)}$ in powers of $s - r$ gives

$$\sum_u \frac{1}{u! \gamma!} \int_0^{\infty} \int_{-\infty}^{\infty} e^{\sqrt{-1}(r-s)\zeta} c_{j\beta}(y, r, 0, \zeta, \mu) \bar{d}\zeta ds \phi^{(\beta)}(y, r)$$

plus a harmless remainder. Here $\beta = (\gamma_1, \dots, \gamma_{m-1}, u)$. Writing \int_0^{∞} as $\int_{-\infty}^{\infty} - \int_{-\infty}^0$, we find the $C_{j\beta}$ and $A_{j\beta}$ as in Lemma 2.2. \square

We continue our development. Let $d_{j\beta} = (-\sqrt{-1})^{|\beta|} \partial_{\xi, \zeta}^\beta d_j$.

Lemma 2.3. $[\text{Op}'(d_j)\phi](y, r) \sim \sum_{\beta} \frac{1}{\beta!} B_{j\beta} \phi^{(\beta)}(y, r, \mu)$ with

$$B_{j\beta}(y, r, \mu) = \int_0^{\infty} \int_{-\infty}^{\infty} e^{-\sqrt{-1}s\zeta} d_{j\beta}(y, r, 0, \zeta, \mu) \bar{d}\zeta ds.$$

Furthermore

$$(2.k) \quad [A_{j\beta} + B_{j\beta} + C_{j\beta}](y, 0, \mu) = 0.$$

Proof. The proof is similar to the proof given in Lemma 2.2. We sketch the details as follows. We express

$$[\text{Op}'(d_j)\phi](y, r) = \int \int \int \int e^{\sqrt{-1}(y-\tilde{y}) \cdot \xi - \sqrt{-1}s\zeta} d_j(y, r, \xi, \zeta, \mu) \phi(\tilde{y}, s) d\tilde{y} ds d\bar{\zeta} d\bar{\xi}$$

and expand $\phi(\tilde{y}, s)$ in powers of $(\tilde{y} - y, s - r)$. To establish Equation (2.k), we use Equation (2.g) to see that:

$$\begin{aligned} [A_{j\beta} + B_{j\beta}](y, 0, \mu) &= - \int_{-\infty}^0 \int_{-\infty}^{\infty} e^{-\sqrt{-1}s\zeta} c_{j\beta}(y, 0, 0, \zeta, \mu) d\bar{\zeta} ds \\ &\quad - \int_0^{\infty} \int_{-\infty}^{\infty} e^{-\sqrt{-1}s\zeta} c_{j\beta}(y, 0, 0, \zeta, \mu) d\bar{\zeta} ds \\ &= -c_{j\beta}(y, 0, 0, 0, \mu) = -C_{j\beta}(y, 0, \mu). \end{aligned}$$

The Lemma now follows. \square

Next, we consider $\langle (D_B + \mu^2)^{-1} \phi, \rho \rangle$ as $\mu \rightarrow \infty$ where $\rho(y, r) = \rho_0(y) r^{-\alpha}$. Define

$$(2.1) \quad A_{j\beta}^{\sim}(y, r, t, \mu) := - \int_{-\infty}^{-t} \int_{-\infty}^{\infty} e^{-\sqrt{-1}s\zeta} c_{j\beta}(y, r, 0, \zeta, \mu) d\bar{\zeta} ds.$$

We set $\mu' := \mu/|\mu|$. We then have

$$(2.m) \quad A_{j\beta}(y, r, \mu) = |\mu|^{j-|\beta|} A_{j\beta}^{\sim}(y, r, |\mu|r, \mu').$$

Lemma 2.4. *We have:*

$$\begin{aligned} &\int_0^{\infty} \langle A_{j\beta}(y, r, \mu) \phi^{(\beta)}(y, r), \rho_0(y) \rangle r^{-\alpha} dr \\ &\sim \sum_{k=0}^{\infty} |\mu|^{\alpha-1+j-k-|\beta|} \int_0^{\infty} \frac{1}{k!} \langle \partial_r^k [A_{j\beta}^{\sim}(y, r, t, \mu') \phi^{(\beta)}(y, r)], \rho_0(y) \rangle |_{r=0} t^{k-\alpha} dt. \end{aligned}$$

The remainder after N terms of the expansion is analytic for $\text{Re}(\alpha) < N$ and is $O(|\mu|^{1+j-N-|\beta|})$ uniformly in $|\text{Re}(\alpha)| < N$.

Proof. With a change of variable $|\mu|r = t$, we have

$$\begin{aligned} &\int_0^{\infty} \langle A_{j\beta} \phi^{(\beta)}, \rho_0 \rangle r^{-\alpha} dr \\ &= |\mu|^{\alpha-1+j-|\beta|} \int_0^{\infty} \langle A_{j\beta}^{\sim}(y, t/|\mu|, t, \mu') \phi^{(\beta)}(y, t/|\mu|), \rho_0(y) \rangle t^{-\alpha} dt. \end{aligned}$$

The Lemma follows from a Taylor expansion of $A_{j\beta}^{\sim}(y, r, t, \mu') \phi^{(\beta)}(y, r)$ in powers of r . This expansion is justified as follows. In Equation (2.1), the integral $d\bar{\zeta}$ is $O((1+s)^{-\infty})$ and so are its derivatives in (y, r) in view of Equation (2.f). Hence $A_{j\beta}^{\sim}$ and its derivatives in (y, r) are $O((1+t)^{-\infty})$. \square

For the term with $B_{j\beta} \phi^{(\beta)}$, we define

$$d_{j\beta}^{\sim}(y, r, \xi, s, \mu) = \int_{-\infty}^{\infty} e^{-\sqrt{-1}s\zeta} d_{j\beta}(y, r, \xi, \zeta, \mu) d\bar{\zeta}.$$

One then has that

$$(2.n) \quad B_{j\beta}(y, r, \mu) = \int_0^{\infty} d_{j\beta}^{\sim}(y, r, 0, s, \mu) ds.$$

Lemma 2.5.

$$\begin{aligned} &\int_0^{\infty} \langle B_{j\beta} \phi^{(\beta)}(y, r), \rho_0(y) \rangle r^{-\alpha} dr \\ &\sim \sum_{k=0}^{\infty} |\mu|^{\alpha-1+j-k-|\beta|} \frac{1}{k!} \int_0^{\infty} \int_0^{\infty} \langle d_{j\beta}^{\sim}(y, r, 0, s, \mu') \partial_r^k \phi^{(\beta)}(y, 0), \rho_0(y) \rangle r^{k-\alpha} ds dr. \end{aligned}$$

The remainder after N terms is bounded as in Lemma 2.4.

Proof. From Equation (2.h), noting that $d_{j\beta} = (-\sqrt{-1})^{|\beta|} \partial_{(\xi, \zeta)}^\beta d_j$, we have

$$d_{j\beta}^\sim(y, r/t, t\xi, s/t, t\mu) = t^{j-|\beta|+1} d_{j\beta}^\sim(y, r, \xi, s, \mu).$$

We have from Equation (2.i) for all N that

$$(1 + s^2) r^N d_{j\beta}^\sim(y, r, 0, s, \mu') = O(1).$$

The Lemma now follows from a Taylor expansion of $\phi^{(\beta)}$ in powers of r . \square

Finally, since $c_{j\beta}$ is homogeneous of degree $j - |\beta|$ in (ξ, ζ, μ) , we have from Lemma 2.2 that:

$$\begin{aligned} & \int_0^\infty \langle C_{j\beta}(y, r, \mu) \phi^{(\beta)}(y, r), \rho_0(y) \rangle r^{-\alpha} dr \\ (2.o) \quad &= |\mu|^{j-|\beta|} \int_0^\infty \langle c_{j\beta}(y, r, 0, 0, \mu') \phi^{(\beta)}(y, r), \rho_0(y) \rangle r^{-\alpha} dr. \end{aligned}$$

We may add up all the expansions in Equation (2.o), in Lemma 2.4, and in Lemma 2.5. All terms are analytic for $\text{Re}(\alpha) < 1$ and we extend the expansion by a meromorphic continuation to $\text{Re}(\alpha) < 2$ with a simple pole at $\alpha = 1$. The terms with a singularity at $\alpha = 1$ are

$$\begin{aligned} & |\mu|^{j-|\beta|} \int_0^\infty \langle c_{j\beta}(y, r, 0, 0, \mu') \phi^{(\beta)}(y, r), \rho_0(y) \rangle r^{-\alpha} dr \\ &+ |\mu|^{\alpha-1+j-|\beta|} \int_0^\infty \langle A_{j\beta}^\sim(y, 0, t, \mu') \phi^\beta(y, 0), \rho_0(y) \rangle t^{-\alpha} dt \\ &+ |\mu|^{\alpha-1+j-|\beta|} \int_0^\infty \int_0^\infty \langle d_{j\beta}^\sim(y, r, 0, s, \mu') \phi^{(\beta)}(y, 0), \rho_0(y) \rangle r^{-\alpha} ds dr. \end{aligned}$$

We must now determine the extension to $\alpha = 1$. As previously, when we regularize the interior integral, we investigate

$$(2.p) \quad \frac{c_{j\beta}(y, 0, 0, 0, \mu') + |\mu|^{\alpha-1} \{ A_{j\beta}^\sim(y, 0, 0, \mu') + \int_0^\infty d_{j\beta}^\sim(y, 0, 0, s, \mu') ds \}}{1 - \alpha}.$$

From Equation (2.m) with $|\mu| = 1$, we have $A_{j\beta}^\sim(y, 0, 0, \mu') = A_{j\beta}(y, 0, \mu')$. From Equation (2.n),

$$\int_0^\infty d_{j\beta}^\sim(y, 0, 0, s, \mu') ds = B_{j\beta}(y, 0, \mu').$$

Then from Equation (2.k) and Lemma 2.2, the expression in Equation (2.p) is

$$c_{j\beta}(y, 0, 0, 0, \mu') \frac{1 - |\mu|^{\alpha-1}}{1 - \alpha} \quad \text{for } \alpha \neq 1.$$

Taking the limit as $\alpha \rightarrow 1$ yields

$$c_{j\beta}(y, 0, 0, 0, \mu') \ln |\mu|, \quad \alpha = 1$$

and consequently we have:

Lemma 2.6. *For the parametrix $P_N = \sum_{j=2}^N [\text{Op}(c_j) + \text{Op}'(d_j)]$, for $\phi \in \mathcal{S}(\mathbb{R}_+^m)$, for $\rho(y, r) = \rho_0(y)r^{-\alpha}$, for $\text{Re}(\alpha) < 2$, and for $\alpha \neq 1$, we have*

$$\begin{aligned} \langle P_N \phi, \rho \rangle &\sim \sum_{j, \beta} |\mu|^{j-|\beta|} \frac{1}{\beta!} \int \int_0^\infty \langle c_{j\beta}(y, r, 0, 0, \mu') \phi^{(\beta)}(y, r), \rho_0(y) \rangle r^{-\alpha} dr dy \\ &+ \sum_{j, \beta, k} |\mu|^{\alpha-1+j-|\beta|-k} \frac{1}{\beta! k!} \int \int_0^\infty \partial_r^k \langle A_{j\beta}^\sim \phi^{(\beta)}, \rho_0 \rangle(y, 0, s, \mu') s^{k-\alpha} ds dy \\ &+ \sum_{j, \beta, k} |\mu|^{\alpha-1+j-|\beta|-k} \frac{1}{\beta! k!} \int \int_0^\infty \int_0^\infty d_{j\beta}^\sim(y, r, 0, s, \mu') r^{k-\alpha} dr ds \\ &\quad \cdot \langle \partial_r^k \phi^{(\beta)}(y, 0), \rho_0(y) \rangle dy. \end{aligned}$$

When $\text{Re}(\alpha) \geq 1$, the divergent integrals are regularized as was discussed previously; when $\alpha = 1$ there are additional terms

$$\sum_{j, \beta} |\mu|^{j-|\beta|} \ln |\mu| \frac{1}{\beta!} \int \langle c_{j\beta}(y, 0, 0, 0, \mu') \phi^{(\beta)}(y, 0), \rho_0(y) \rangle dy.$$

The expansion in Lemma 2.6 is valid with $|\mu|$ replaced by μ and μ' replaced by 1 in view of the following result:

Lemma 2.7. *If $f(\mu)$ is holomorphic in a sector \mathcal{S} which contains the positive real axis and if $f(\mu) = |\mu|^j g(\mu') + o(|\mu|^j)$, then $f(\mu) = g(1)\mu^j + o(|\mu|^j)$.*

Proof. Let C be any closed curve in the sector \mathcal{S} . Then for $t > 0$,

$$\begin{aligned} 0 &= t^{-j-1} \int_{tC} f(\mu) d\mu = t^{-j-1} \int_{tC} \{ |\mu|^j g(\mu') + o(|\mu|^j) \} d\mu \\ &= \int_C \{ |z|^j g(z') + o(1) \} dz. \end{aligned}$$

Let $t \rightarrow \infty$. By Morera's theorem, $|z|^j g(z')$ is holomorphic in \mathcal{S} so it equals $z^j g(1)$, the holomorphic extension of its value on $\{z > 0\}$. \square

2.3. The case of a manifold M . The proof of Theorem 2.1 now follows by standard arguments [13, 11] involving a parametrix P_N on the manifold M constructed from the Euclidean parametrices as in Lemma 2.6. In showing that

$$(2.q) \quad \langle P_N \phi - (D_B + \mu^2)^{-1} \phi, \rho \rangle = O(\mu^{-K})$$

for large K , we need to deal with the singularity $r^{-\alpha}$ in the specific heat. To this end, let $R_N(y, r, \tilde{y}, s, \mu)$ be the kernel of $(D_B + \mu^2)^{-1} - P_N$. Then R_N and its first derivatives are $O(\mu^{-K})$ for large K . Moreover, by construction, the kernel of P_N is zero when $r = 0$ so the same is true of R_N . It follows that R_N is $O(r\mu^{-K})$ for large K and Equation (2.q) follows.

3. HEAT CONTENT ASYMPTOTICS FOR DIRICHLET BOUNDARY CONDITIONS

We adopt the notation of Theorem 1.4 throughout this section. Let $B = B_{\mathcal{D}}$ define Dirichlet boundary conditions. We begin by using dimensional analysis to express the invariants $\beta_{k, \alpha}^{\partial M}$ in terms of a Weyl basis of invariants which is formed by contracting indices.

Lemma 3.1. *There exist universal constants ε_α^i so that:*

$$\begin{aligned} \int_{\partial M} \beta_{0, \alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy &= \int_{\partial M} \varepsilon_\alpha^0 \langle \phi_0, \rho_0 \rangle dy, \\ \int_{\partial M} \beta_{1, \alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy &= \int_{\partial M} \{ \varepsilon_\alpha^1 \langle \phi_1, \rho_0 \rangle + \varepsilon_\alpha^2 \langle L_{aa} \phi_0, \rho_0 \rangle + \varepsilon_\alpha^3 \langle \phi_0, \rho_1 \rangle \}, \\ \int_{\partial M} \beta_{2, \alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy &= \int_{\partial M} \{ \varepsilon_\alpha^4 \langle \phi_2, \rho_0 \rangle + \varepsilon_\alpha^5 \langle L_{aa} \phi_1, \rho_0 \rangle + \varepsilon_\alpha^6 \langle E \phi_0, \rho_0 \rangle \\ &\quad + \varepsilon_\alpha^7 \langle \phi_0, \rho_2 \rangle + \varepsilon_\alpha^8 \langle L_{aa} \phi_0, \rho_1 \rangle + \varepsilon_\alpha^9 \langle \text{Ric}_{mm} \phi_0, \rho_0 \rangle + \varepsilon_\alpha^{10} \langle L_{aa} L_{bb} \phi_0, \rho_0 \rangle \} \end{aligned}$$

$+\varepsilon_\alpha^{11}\langle L_{ab}L_{ab}\phi_0, \rho_0\rangle + \varepsilon_\alpha^{12}\langle \phi_{0;a}, \rho_{0;a}\rangle + \varepsilon_\alpha^{13}\langle \tau\phi_0, \rho_0\rangle + \varepsilon_\alpha^{14}\langle \phi_1, \rho_1\rangle\}dy$,
 where τ is the scalar curvature.

Proof. Let $c > 0$. We consider a metric $g_c := c^2g$. We then have for $\alpha \neq 1$ (see the paragraph after (3.b) for the case $\alpha = 1$):

$$(3.a) \quad \begin{aligned} dx_c &:= c^m dx, & dy_c &:= c^{m-1} dy, & D_c &:= c^{-2} D, \\ r_c &:= cr, & \partial_{r_c} &= c^{-1} \partial_r, & \phi_{i,c} &= c^{\alpha-i} \phi_i, \\ \mathcal{I}_{\text{Reg},c} &= c^m \mathcal{I}_{\text{Reg}}, & \rho_{i,c} &= c^{-i} \rho_i. \end{aligned}$$

We then compute:

$$\begin{aligned} \beta(\phi, \rho, D_c, B_{\mathcal{D}})(t) &= \int_M \langle e^{-tD_{c,B}} \phi, \rho \rangle dx_c \\ &= c^m \int_M \langle e^{c^{-2}tD_B} \phi, \rho \rangle dx = c^m \beta(\phi, \rho, D, B_{\mathcal{D}})(c^{-2}t). \end{aligned}$$

We take $\alpha \notin \mathbb{Z}$ and then continue analytically to the integer values with $\alpha \neq 1$. The interior and boundary terms then decouple so we may conclude:

$$\begin{aligned} &\sum_{k=0}^{\infty} t^{(1+k-\alpha)/2} \int_{\partial M} \sum_{i+j \leq k} \beta_{k,\alpha,i,j}^{\partial M}(c^{\alpha-i} \phi_i, c^{-j} \rho_j, c^{-2} D, B_{\mathcal{D}}) c^{m-1} dy \\ &= c^m \sum_{k=0}^{\infty} c^{\alpha-k-1} t^{(1+k-\alpha)/2} \int_{\partial M} \sum_{i+j \leq k} \beta_{k,\alpha,i,j}^{\partial M}(\phi_i, \rho_j, D, B_{\mathcal{D}}) dy. \end{aligned}$$

Equating powers of t in the asymptotic series and simplifying yields:

$$\int_{\partial M} \beta_{k,\alpha,i,j}^{\partial M}(\phi_i, \rho_j, c^{-2} D, B_{\mathcal{D}}) dy = c^{i+j-k} \int_{\partial M} \beta_{k,\alpha,i,j}^{\partial M}(\phi_i, \rho_j, D, B_{\mathcal{D}}) dy.$$

Studying relations of this kind is by now quite standard and we refer to [9] for further details. We conclude that $\beta_{k,\alpha,i,j}^{\partial M}$ is homogeneous of weighted degree k in the jets of ϕ_i and of ρ_j . We now use Weyl's theory of invariants to write down a spanning set for the invariants which arise in this way. There is some indeterminacy in these invariants as we can always integrate by parts to eliminate tangential divergence terms. For example, we have

$$(3.b) \quad \int_{\partial M} \langle \phi_{0;aa}, \rho_0 \rangle dy = \int_{\partial M} \langle \phi_0, \rho_{0;aa} \rangle dy = - \int_{\partial M} \langle \phi_{0;a}, \rho_{;a} \rangle dy.$$

For this reason, we have eliminated the first two invariants from the formula in Lemma 3.1. This completes the proof of Lemma 3.1 for $\alpha \neq 1$.

If $\alpha = 1$, the argument is different. In this instance, the regularizing term is given by:

$$\begin{aligned} \ln(\varepsilon_c) \int_{\partial M} \langle \phi_{0,c}, \rho_{0,c} \rangle dy_c &= \ln(c\varepsilon) \int_{\partial M} c \langle \phi_0, \rho_0 \rangle c^{m-1} dy \\ &= c^m \{ \ln(\varepsilon) + \ln(c) \} \int_{\partial M} \langle \phi_0, \rho_0 \rangle dy. \end{aligned}$$

This then yields the modified relation:

$$(3.c) \quad \mathcal{I}_{\text{Reg},c}(\phi, \rho) = c^m \mathcal{I}_{\text{Reg}}(\phi, \rho) + \ln(c) c^m \int_{\partial M} \langle \phi_0, \rho_0 \rangle dy.$$

Recall the notation of Theorem 1.4. One has that:

$$\begin{aligned}
& \beta(\phi, \rho, c^{-2}D, B_{\mathcal{D}})(t) \sim \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \mathcal{I}_{\text{Reg},c}(\phi, c^{-2n} \tilde{D}^n \rho) \\
& + \sum_{k=0}^{\infty} t^{k/2} \ln(t) \int_{\partial M} \check{\beta}_k(\phi, \rho, c^{-2}D, B_{\mathcal{D}}) c^{m-1} dy \\
& + \sum_{k=0}^{\infty} \sum_{i+j \leq k} t^{k/2} \int_{\partial M} \beta_{k,1,i,j}^{\partial M} (c^{1-i} \phi_i, c^{-j} \rho_j, c^{-2}D, B_{\mathcal{D}}) c^{m-1} dy \\
& = c^m \beta(\phi, \rho, D, B_{\mathcal{D}})(c^{-2}t) \sim \sum_{n=0}^{\infty} c^{m-2n} \frac{(-t)^n}{n!} \mathcal{I}_{\text{Reg}}(\phi, \tilde{D}^n \rho) \\
& + \sum_{k=0}^{\infty} c^{m-k} t^{k/2} (\ln(t) - 2 \ln(c)) \int_{\partial M} \check{\beta}_k(\phi, \rho, D, B_{\mathcal{D}}) dy \\
& + \sum_{k=0}^{\infty} \sum_{i+j \leq k} c^{m-k} t^{k/2} \int_{\partial M} \beta_{k,1,i,j}^{\partial M} (\phi_i, \rho_j, D, B_{\mathcal{D}}) dy.
\end{aligned}$$

Equating terms in the asymptotic expansion then yields:

$$\begin{aligned}
(3.d) \quad & \int_{\partial M} \beta_{k,1,i,j}^{\partial M} (\phi_i, \rho_j, c^{-2}D, B_{\mathcal{D}}) dy = c^{i+j-k} \int_{\partial M} \beta_{k,1,i,j}^{\partial M} (\phi_i, \rho_j, D, B_{\mathcal{D}}) dy, \\
& \frac{(-1)^n}{n!} \mathcal{I}_{\text{Reg},c}(\phi, \tilde{D}^n \rho) = c^m \left\{ \mathcal{I}_{\text{Reg}}(\phi, \tilde{D}^n \rho) - 2 \ln(c) \int_{\partial M} \check{\beta}_{2n}(\phi, \rho, D, B_{\mathcal{D}}) dy \right\}, \\
& 0 = -2 \ln(c) \int_{\partial M} \check{\beta}_{2n+1}(\phi, \rho, D, B_{\mathcal{D}}) dy.
\end{aligned}$$

The weighted homogeneity of $\beta_{k,1,i,j}^{\partial M}$ now follows and completes the proof of Lemma 3.1. Theorem 1.4 (4) follows from Equations (3.c) and (3.d) \square

We shall prove Assertions (1) and (2) of Theorem 1.5 by evaluating the normalizing constants in Lemma 3.1. We begin by establishing some product formulas:

Lemma 3.2. *Suppose that $M = M_1 \times M_2$, that $g_M = g_{M_1} + g_{M_2}$, that $\partial M_1 = \emptyset$, and that $D_M = D_{M_1} + D_{M_2}$ where D_{M_1} and D_{M_2} are scalar operators of Laplace type on M_1 and on M_2 , respectively. Suppose that $\phi_M = \phi_{M_1} \phi_{M_2}$ and $\rho_M = \rho_{M_1} \rho_{M_2}$ decompose similarly. Then*

- (1) $\beta(\phi_M, \rho_M, D_M, B_{\mathcal{D}})(t) = \beta(\phi_{M_1}, \rho_{M_1}, D_{M_1})(t) \cdot \beta(\phi_{M_2}, \rho_{M_2}, D_{M_2}, B_{\mathcal{D}})(t).$
- (2) $\int_{\partial M} \beta_{k,\alpha}^{\partial M}(\phi_M, \rho_M, D_M, B_{\mathcal{D}}) dy$
 $= \sum_{2n+j=k} \frac{(-1)^n}{n!} \int_{M_1} \langle \phi_{M_1}, (\tilde{D}_{M_1})^n \rho_{M_1} \rangle dx_{M_1}$
 $\times \int_{\partial M_2} \beta_{j,\alpha}^{\partial M_2}(\phi_{M_2}, \rho_{M_2}, D_{M_2}, B_{\mathcal{D}}) dy_{M_2}.$
- (3) *The universal constants ε_{α}^i are dimension free.*
- (4) $\varepsilon_{\alpha}^6 = \varepsilon_{\alpha}^0$, $\varepsilon_{\alpha}^{13} = 0$, and $\varepsilon_{\alpha}^{12} = -\varepsilon_{\alpha}^0$.

Proof. Assertion (1) follows from the identity $e^{-tD_{M,B}} = e^{-tD_{M_1}} e^{-tD_{M_2,B}}$ and Assertion (2) follows from Assertion (1). If we take $M_1 = S^1$, $D_{M_1} = -\partial_{\theta}^2$, $\phi_{M_1} = 1$, and $\rho_{M_1} = 1$, we have that $\beta(\phi_{M_1}, \rho_{M_1}, D_{M_1})(t) = 2\pi$. This then yields the identity

$$\int_{\partial M} \beta_{k,\alpha}^{\partial M}(\phi_{M_2}, \rho_{M_2}, D, B_{\mathcal{D}}) dy = 2\pi \int_{\partial M_2} \beta_{k,\alpha}^{\partial M_2}(\phi_{M_2}, \rho_{M_2}, D_{M_2}, B_{\mathcal{D}}) dy_2.$$

Assertion (3) now follows.

We take $M_2 = [0, 1]$ and $D_2 = -\partial_r^2$. We take

$$\begin{aligned}
\phi_{M_2} &= \rho_{M_2} = 0 \quad \text{near } r = 1, \\
\rho_{M_2} &= 1 \quad \text{and } \phi_{M_2} = r^{-\alpha} \quad \text{near } r = 0.
\end{aligned}$$

Since the structures on M_2 are flat,

$$\beta_k^{\partial M_2}(\phi_{M_2}, \rho_{M_2}, D_{M_2}, B_{\mathcal{D}})(r) = \begin{cases} 0 & \text{if } r = 1 \text{ and } k \geq 0, \\ 0 & \text{if } r = 0 \text{ and } k > 0, \\ \varepsilon_\alpha^0 & \text{if } r = 0 \text{ and } k = 0. \end{cases}$$

As the second fundamental form vanishes, the distinction between ‘;’ and ‘:’ disappears and we may use Equation (1.c) to see that $\tilde{D}_1 \rho_{M_1} = -(\rho_{;aa} + \tilde{E}\rho)$. Theorem 1.4 then implies

$$\beta_2(\phi_{M_1}, \rho_{M_1}, D_{M_1}) = \int_{\partial M} \langle \phi_{M_1}, \rho_{M_1;aa} + \tilde{E}\rho_{M_1} \rangle dx_1.$$

We may therefore use Assertion (2) to see

$$\int_{\partial M} \beta_{2,\alpha}^{\partial M}(\phi_M, \rho_M, D_M, B_{\mathcal{D}}) dy = \varepsilon_\alpha^0 \int_{M_1} \langle \phi_{M_1}, \rho_{M_1;aa} + \tilde{E}\rho_{M_1} \rangle dx_1$$

Assertion (4) now follows from this identity. \square

Next, we evaluate the universal constants ε_α^0 .

Lemma 3.3.

- (1) If $\alpha \neq 1$, then $\varepsilon_\alpha^0 = \pi^{-1/2} 2^{1-\alpha} \Gamma\left(\frac{2-\alpha}{2}\right) (\alpha-1)^{-1}$.
- (2) Let γ be Euler’s constant. Then $\varepsilon_1^0 = \frac{\gamma}{2}$.

Proof. The proof follows from (2.d). We note that Assertion (1) also follows for $1 < \alpha < 2$ by the special case calculation in [2]. Assertion (1) then follows for $\alpha \neq 1$ by analytic continuation.

To study the case $\alpha = 1$ by a special case calculation we let $M = [0, \infty)$, let $\Theta = 1$ on $[0, \varepsilon]$ and with compact support in $[0, 1)$, let $\phi = r^{-1}\Theta(r)$, let $\rho = 1$, let $D = -\partial_r^2$, and let γ be Euler’s constant. As is usual, we work dually and compute $\beta(\rho, \phi, D, B_{\mathcal{D}})(t)$. The halfspace solution of the heat equation with constant initial temperature is given by:

$$u(r; t) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{r}{2\sqrt{t}}} e^{-s^2} ds.$$

Consequently, we may compute:

$$\begin{aligned} \beta(\rho, \phi, D, B_{\mathcal{D}})(t) &= \frac{2}{\sqrt{\pi}} \int_0^1 \int_0^{\frac{r}{2\sqrt{t}}} e^{-s^2} r^{-1} \Theta(r) ds dr \\ &= \frac{2}{\sqrt{\pi}} \int_0^1 \ln(r) \partial_r \left\{ \Theta(r) \int_0^{\frac{r}{2\sqrt{t}}} e^{-s^2} ds \right\} dr \\ &= -\frac{1}{\sqrt{\pi t}} \int_0^1 \ln(r) \Theta(r) e^{-\frac{r^2}{4t}} dt - \frac{2}{\sqrt{\pi}} \int_\varepsilon^1 \int_0^{\frac{r}{2\sqrt{t}}} e^{-s^2} \ln(r) \Theta'(r) ds dr \\ &= B_1 + B_2, \end{aligned}$$

where

$$\begin{aligned} B_1 &= -\frac{1}{\sqrt{\pi t}} \left\{ \int_0^1 \ln(r) e^{-\frac{r^2}{4t}} dr + \int_\varepsilon^1 \ln(r) (\Theta(r) - 1) e^{-\frac{r^2}{4t}} dr \right\} \\ &= -\frac{1}{\sqrt{\pi t}} \int_0^\infty \ln(r) e^{-\frac{r^2}{4t}} dr + O(e^{-\frac{\varepsilon^2}{8t}}) \\ &= -\frac{2}{\sqrt{\pi}} \left\{ \int_0^\infty \ln(r) e^{-r^2} dr + \int_0^\infty \ln(2\sqrt{t}) e^{-r^2} dr \right\} + O(e^{-\frac{\varepsilon^2}{8t}}) \\ &= -\frac{1}{2} \ln(t) + \left\{ -\ln(2) - \frac{2}{\sqrt{\pi}} \int_0^\infty \ln(r) e^{-r^2} dr \right\} + O(e^{-\frac{\varepsilon^2}{8t}}), \\ B_2 &= -\frac{2}{\sqrt{\pi}} \int_\varepsilon^1 \ln(r) \Theta'(r) \left\{ \int_0^\infty e^{-s^2} ds - \int_{\frac{r}{2\sqrt{t}}}^\infty e^{-s^2} ds \right\} dr \end{aligned}$$

$$\begin{aligned}
&= - \int_{\varepsilon}^1 \ln(r) \Theta'(r) dr + O(e^{-\frac{\varepsilon^2}{8t}}) \\
&= - \ln(r) \Theta(r) \Big|_{\varepsilon}^1 + \int_{\varepsilon}^1 r^{-1} \Theta(r) dr + O(e^{-\frac{\varepsilon^2}{8t}}) \\
&= \ln(\varepsilon) + \int_{\varepsilon}^1 r^{-1} \Theta(r) dr + O(e^{-\frac{\varepsilon^2}{8t}}).
\end{aligned}$$

This then yields the expression

$$\begin{aligned}
&\beta(\rho, \phi, D, B_{\mathcal{D}})(t) \\
&= \frac{1}{2} \ln\left(\frac{\varepsilon^2}{t}\right) - \ln(2) - \frac{2}{\sqrt{\pi}} \int_0^{\infty} \ln(s) e^{-s^2} ds + \int_{\varepsilon}^1 r^{-1} \Theta(r) dr + O(e^{-\frac{\varepsilon^2}{8t}}).
\end{aligned}$$

Since ϕ is compactly supported in $[0, 1)$, the heat content for corresponding problem on the interval $[0, 1]$ is the same as for $[0, \infty)$ up to $O(t^{\infty})$. Assertion (2) now follows. \square

We continue our study by index shifting:

Lemma 3.4.

(1) Assume $\nabla_{\partial_r} \Phi = 0$ on $\mathcal{C}_{\varepsilon}$. Set $\phi := \chi(r) \Phi(y) r^{i_0 - \gamma}$. Then

$$\int_{\partial M} \beta_{k, \gamma, i_0, j}^{\partial M}(\Phi, \rho_j, D, B_{\mathcal{D}}) dy = \int_{\partial M} \beta_{k-i_0, \gamma-i_0, 0, j}^{\partial M}(\phi, \rho_j, D, B_{\mathcal{D}}) dy.$$

(2) If $\alpha \neq 1$, then $\varepsilon_{\alpha}^1 = \varepsilon_{\alpha-1}^0$, $\varepsilon_{\alpha}^4 = \varepsilon_{\alpha-2}^0$, $\varepsilon_{\alpha}^5 = \varepsilon_{\alpha-1}^2$, and $\varepsilon_{\alpha}^{14} = \varepsilon_{\alpha-1}^3$.

(3) $\varepsilon_1^1 = -\frac{2}{\sqrt{\pi}}$, $\varepsilon_1^4 = -1$, $\varepsilon_1^5 = \frac{1}{2}$, and $\varepsilon_1^{14} = 0$.

Proof. Set $\phi := \chi(r) \Phi(y) r^{i_0 - \gamma}$. By Theorem 1.4 with $\alpha = \gamma$ and with $\alpha = \gamma - i_0$,

$$\begin{aligned}
&\sum_{i_0+j \leq k} t^{(1+k-\gamma)/2} \int_{\partial M} \beta_{k, \gamma, i_0, j}(\Phi, \rho_j, D, B_{\mathcal{D}}) dy \\
&\sim \sum_{j \leq \ell} t^{(1+\ell-(\gamma-i_0))/2} \int_{\partial M} \beta_{\ell, \gamma-i_0, 0, j}(\phi, \rho_j, D, B_{\mathcal{D}}) dy.
\end{aligned}$$

We set $k = \ell + i_0$ and equate powers of t to establish Assertion (1); Assertion (2) then follows from Lemma 3.1 and Assertion (3) from Theorem 1.3. \square

We continue our study with a functorial property that exploits the fact that we are working in a very general context; we are no longer working with the scalar Laplacian! Even if one were only interested in the scalar Laplacian, it would be necessary to consider general operators of Laplace type in order to use this functorial property! Let $\Theta = 1$ on $[0, \frac{1}{2}]$ and with compact support in $[0, 1)$.

Lemma 3.5. Let \mathbb{T}^{m-1} be the torus with periodic parameters (y_1, \dots, y_{m-1}) . Let $M = \mathbb{T}^{m-1} \times [0, 1]$. Let $f_a \in C^{\infty}([0, 1])$ satisfy $f_a(0) = 0$ and $f_a \equiv 0$ near $r = 1$. Let $\delta_a \in \mathbb{R}$. Set

$$\begin{aligned}
ds_M^2 &= \sum_a e^{2f_a(r)} dy_a \circ dy_a + dr \circ dr, & \rho &:= e^{-\sum_a f_a(r)}, \\
D_M &:= - \sum_a e^{-2f_a(r)} (\partial_{y_a}^2 + \delta_a \partial_{y_a}) - \partial_r^2, & \phi &:= \Theta(r) r^{-\alpha}.
\end{aligned}$$

(1) If $k > 0$, then $\int_{\partial M} \beta_{k, \alpha}^{\partial M}(\phi, \rho, D_M, B_{\mathcal{D}}) dy = 0$.

(2) $-\frac{1}{2} \varepsilon_{\alpha}^1 - \varepsilon_{\alpha}^2 = 0$.

(3) $-\frac{1}{4} (\varepsilon_{\alpha}^6 + \varepsilon_{\alpha}^{12}) = 0$.

(4) $-\frac{1}{4} \varepsilon_{\alpha}^4 + \frac{1}{2} \varepsilon_{\alpha}^6 - \frac{1}{4} \varepsilon_{\alpha}^7 - \varepsilon_{\alpha}^9 = 0$.

(5) $\frac{1}{8} \varepsilon_{\alpha}^4 + \frac{1}{2} \varepsilon_{\alpha}^5 + \frac{1}{4} \varepsilon_{\alpha}^6 + \frac{1}{8} \varepsilon_{\alpha}^7 + \frac{1}{2} \varepsilon_{\alpha}^8 + \varepsilon_{\alpha}^{10} = 0$.

(6) $-\varepsilon_{\alpha}^9 + \varepsilon_{\alpha}^{11} = 0$.

Proof. We use $-\partial_r^2$ on $[0, 1]$ and D_M on M . Since ϕ vanishes near $r = 1$, this boundary component plays no role. Let $u_B(r; t)$ be the solution of the heat equation on $[0, 1]$ with Dirichlet boundary conditions. Since the problem decouples, $u_B(r; t)$ is also the solution of the heat equation on M with Dirichlet boundary conditions. The Riemannian measure

$$dx = \sqrt{\det g_{ij}} dy dr = e^{\sum_a f_a} dy dr.$$

As $\rho = e^{-\sum_a f_a}$, $\rho dx = dy dr$. We suppose $\alpha \neq 1$. Since $\text{vol}(\mathbb{T}^{m-1}) = (2\pi)^{m-1}$,

$$\begin{aligned} \beta(\phi, \rho, D, B_{\mathcal{D}})(t) &= \int u_B(r; t) \rho dx = (2\pi)^{m-1} \int_0^1 u_B(r; t) dr \\ &= (2\pi)^{m-1} \beta(\phi, 1, -\partial_r^2)(t) = (2\pi)^{m-1} (\mathcal{I}_{\text{Reg}}(\phi, 1) + \varepsilon_\alpha^0) + O(t^n) \end{aligned}$$

for any n since the structures are flat on the interval. Note that $\tilde{D}\rho = 0$. Assertion (1) now follows. If $\alpha = 1$, the computation must be modified to take the \ln term into account; this does not affect the computation of $\beta_k^{\partial M}$ for $k > 1$ and the desired conclusion follows similarly.

To apply Assertion (1), we must determine the relevant tensors. The formalism of Lemma 1.1 is crucial at this point as the connection defined by the operator D_M is no longer flat. We have:

$$\begin{aligned} \omega_a &= \frac{1}{2} \delta_a, & \tilde{\omega}_a &= -\omega_a = -\frac{1}{2} \delta_a, \\ \omega_m &= -\frac{1}{2} \sum_a f'_a, & \tilde{\omega}_m &= -\omega_m = \frac{1}{2} \sum_a f'_a. \end{aligned}$$

We compute:

$$\begin{aligned} \phi_0 &= 1, \\ \phi_1 &= \{\nabla_{\partial r}(r^\alpha \phi)\}|_{\partial M} = \{(\partial_r - \frac{1}{2} \sum_a f'_a)(1)\}|_{\partial M} = -\frac{1}{2} \sum_a f'_a(0), \\ \phi_2 &= \frac{1}{2} \{(\nabla_{\partial r})^2(r^\alpha \phi)\}|_{\partial M} = \frac{1}{2} \{(\partial_r - \frac{1}{2} \sum_a f'_a)^2(1)\}|_{\partial M} \\ &= \frac{1}{8} (\sum_a f'_a(0))^2 - \frac{1}{4} \sum_a f''_a(0), \\ \rho_0 &= 1, \\ \rho_1 &= \{\tilde{\nabla}_{\partial r}(\rho)\}|_{\partial M} = \{(\partial_r + \frac{1}{2} \sum_a f'_a)(e^{-\sum_a f_a})\}|_{\partial M} = -\frac{1}{2} \sum_a f'_a(0), \\ \rho_2 &= \frac{1}{2} \{(\tilde{\nabla}_{\partial r})^2 \rho\}|_{\partial M} = \frac{1}{2} \{(\partial_r + \frac{1}{2} \sum_a f'_a)^2(e^{-\sum_a f_a})\}|_{\partial M} \\ &= \frac{1}{8} (\sum_a f'_a(0))^2 - \frac{1}{4} \sum_a f''_a(0). \end{aligned} \tag{3.e}$$

It is straightforward to compute that we have the following relations when $r = 0$; we refer to Lemma 2.3.7 [9] for further details:

$$\begin{aligned} E &= \frac{1}{2} \sum_a f''_a + \frac{1}{4} (\sum_a f'_a)^2 - \frac{1}{4} \sum_a \delta_a^2, & L_{aa} &= -\sum_a f'_a, \\ \text{Ric}_{mm} &= -\sum_a ((f'_a)^2 + f''_a), & L_{aa} L_{bb} &= (\sum_a f'_a)^2, & L_{ab} L_{ab} &= \sum_a (f'_a)^2. \end{aligned} \tag{3.f}$$

Considering the term $\sum_a f'_a$ in $\beta_{1,\alpha}^{\partial M}$ yields Assertion (2), considering the term $\sum_a \delta_a^2$ in $\beta_{2,\alpha}^{\partial M}$ yields Assertion (3), considering the term $\sum_a f''_a$ in $\beta_{2,\alpha}^{\partial M}$ yields Assertion (4), considering the term $(\sum_a f'_a)^2$ in $\beta_{2,\alpha}^{\partial M}$ yields Assertion (5), and considering the term $\sum_a (f'_a)^2$ in $\beta_{2,\alpha}^{\partial M}$ yields Assertion (6). \square

We continue our discussion:

Lemma 3.6.

- (1) If $\rho_0 = 0$, then $\partial_t \beta(\phi, \rho, D, B_{\mathcal{D}})(t) = -\beta(\phi, \tilde{D}\rho, D, B_{\mathcal{D}})(t)$.
- (2) $\varepsilon_\alpha^3 = 0$.
- (3) If $\alpha \neq 1$, $\varepsilon_\alpha^7 = \frac{4}{3-\alpha} \varepsilon_\alpha^0$, $\varepsilon_\alpha^8 = -\frac{2}{3-\alpha} \varepsilon_\alpha^0$, and $\varepsilon_\alpha^{14} = 0$.
- (4) $\varepsilon_1^7 = 2\varepsilon_1^0 + 1$, $\varepsilon_1^8 = -\varepsilon_1^0 - \frac{1}{2}$, and $\varepsilon_\alpha^{14} = 0$.

Proof. Assume $\rho_0 = 0$. By Equation (1.c) we have

$$(3.g) \quad -(\tilde{D}\rho)_0 = 2\rho_2 - L_{aa}\rho_1.$$

We compute that:

$$\begin{aligned}\partial_t \beta(\phi, \rho, D, B_{\mathcal{D}})(t) &= -\langle D e^{-tD_B} \phi, \rho \rangle = -\langle e^{-tD_B} \phi, \tilde{D} \rho \rangle \\ &= -\beta(\phi, \tilde{D} \rho, D, B_{\mathcal{D}})(t),\end{aligned}$$

where the middle equality is justified as $\rho_0 = 0$. This proves Assertion (1). We use Assertion (1) to see

$$\int_{\partial M} \beta_{1,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy = 0.$$

Since there is no restriction on ρ_1 , we conclude that $\varepsilon_{\alpha}^3 = 0$ which establishes Assertion (2). Furthermore, if $\alpha \neq 1$, we may conclude

$$(3.h) \quad \frac{1+k-\alpha}{2} \int_{\partial M} \beta_{k,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{D}}) dy = - \int_{\partial M} \beta_{k-2,\alpha}^{\partial M}(\phi, \tilde{D} \rho, D, B_{\mathcal{D}}) dy.$$

We set $k = 2$ to see that

$$\begin{aligned}& - \int_{\partial M} \varepsilon_{\alpha}^0 \langle \phi_0, (\tilde{D} \rho)_0 \rangle \\ &= \frac{3-\alpha}{2} \int_{\partial M} \{ \varepsilon_{\alpha}^7 \langle \phi_0, \rho_2 \rangle + \varepsilon_{\alpha}^8 \langle L_{aa} \phi_0, \rho_1 \rangle + \varepsilon_{\alpha}^{14} \langle \phi_1, \rho_1 \rangle \} dy.\end{aligned}$$

Assertion (3) now follows from Equation (3.g). If $\alpha = 1$, we have:

$$\begin{aligned}& - \int_{\partial M} \varepsilon_1^0 \langle \phi_0, (\tilde{D} \rho)_0 \rangle \\ &= \int_{\partial M} \left\{ \frac{1}{2} \langle \phi_0, (\tilde{D} \rho)_0 \rangle + \varepsilon_1^7 \langle \phi_0, \rho_2 \rangle + \varepsilon_1^8 \langle L_{aa} \phi_0, \rho_1 \rangle + \varepsilon_1^{14} \langle \phi_1, \rho_1 \rangle \right\} dy.\end{aligned}$$

Assertion (4) now follows. \square

Assertions (1) and (2) of Theorem 1.5 will follow from Lemma 3.2 and from the following result:

Lemma 3.7.

(1) Suppose that $\alpha \neq 1$. Then:

- (a) $\varepsilon_{\alpha}^0 = c_{\alpha}$.
- (b) $\varepsilon_{\alpha}^1 = c_{\alpha-1}$, $\varepsilon_{\alpha}^2 = -\frac{1}{2}c_{\alpha-1}$, and $\varepsilon_{\alpha}^3 = 0$.
- (c) $\varepsilon_{\alpha}^4 = c_{\alpha-2}$ and $\varepsilon_{\alpha}^5 = -\frac{1}{2}c_{\alpha-2}$.
- (d) $\varepsilon_{\alpha}^6 = -\frac{\alpha-3}{2(\alpha-1)(\alpha-2)}c_{\alpha-2}$, $\varepsilon_{\alpha}^7 = \frac{2}{(\alpha-1)(\alpha-2)}c_{\alpha-2}$ and $\varepsilon_{\alpha}^8 = -\frac{1}{(\alpha-1)(\alpha-2)}c_{\alpha-2}$.
- (e) $\varepsilon_{\alpha}^9 = -\frac{\alpha-1}{4(\alpha-2)}c_{\alpha-2}$ and $\varepsilon_{\alpha}^{10} = \frac{\alpha-1}{8(\alpha-2)}c_{\alpha-2}$.
- (f) $\varepsilon_{\alpha}^{11} = -\frac{\alpha-1}{4(\alpha-2)}c_{\alpha-2}$, $\varepsilon_{\alpha}^{12} = \frac{\alpha-3}{2(\alpha-1)(\alpha-2)}c_{\alpha-2}$, $\varepsilon_{\alpha}^{13} = 0$, and $\varepsilon_{\alpha}^{14} = 0$.

(2) Let $\alpha = 1$. Then:

- (a) $\varepsilon_1^0 = \frac{1}{2}\gamma$.
- (b) $\varepsilon_1^1 = -\frac{2}{\sqrt{\pi}}$, $\varepsilon_1^2 = \frac{1}{\sqrt{\pi}}$, and $\varepsilon_1^3 = 0$.
- (c) $\varepsilon_1^4 = -1$, $\varepsilon_1^5 = \frac{1}{2}$.
- (d) $\varepsilon_1^6 = \frac{1}{2}\gamma$, $\varepsilon_1^7 = \gamma + 1$, and $\varepsilon_1^8 = -\frac{1}{2}\gamma - \frac{1}{2}$.
- (e) $\varepsilon_1^9 = 0$ and $\varepsilon_1^{10} = 0$.
- (f) $\varepsilon_1^{11} = 0$, $\varepsilon_1^{12} = \frac{1}{2}\gamma$, $\varepsilon_1^{13} = 0$, and $\varepsilon_1^{14} = 0$.

Proof. Let $\alpha \neq 1$. Assertions (1a) and (2a) follow from Lemma 3.3. Assertions (1b) and (2b) follow from Lemmas 3.4, 3.5, and 3.6. Assertions (1c) and (2c) follow from Assertion (1b) and from Lemma 3.4. Because $s\Gamma(s) = \Gamma(s+1)$, we have:

$$c_{\alpha} = -\frac{\alpha-3}{2(\alpha-1)(\alpha-2)}c_{\alpha-2}.$$

Assertions (1d) and (2d) now follow from Lemmas 3.2 and 3.6. We use Lemma 3.5 to establish Assertions (1e) and (2e) by computing for $\alpha \neq 1$ that

$$\begin{aligned}\varepsilon_\alpha^9 &= -\frac{1}{4}\varepsilon_\alpha^4 + \frac{1}{2}\varepsilon_\alpha^6 - \frac{1}{4}\varepsilon_\alpha^7 \\ &= c_{\alpha-2}\left(-\frac{1}{4} - \frac{1}{4}\frac{\alpha-3}{(\alpha-1)(\alpha-2)} - \frac{1}{4}\frac{2}{(\alpha-1)(\alpha-2)}\right) = -\frac{\alpha-1}{4(\alpha-2)}c_{\alpha-2}, \\ \varepsilon_\alpha^{10} &= -\frac{1}{8}\varepsilon_\alpha^4 - \frac{1}{2}\varepsilon_\alpha^5 - \frac{1}{4}\varepsilon_\alpha^6 - \frac{1}{8}\varepsilon_\alpha^7 - \frac{1}{2}\varepsilon_\alpha^8 \\ &= c_{\alpha-2}\left(-\frac{1}{8} + \frac{1}{4} + \frac{1}{8}\frac{\alpha-3}{(\alpha-1)(\alpha-2)} - \frac{1}{4}\frac{1}{(\alpha-1)(\alpha-2)} + \frac{1}{2}\frac{1}{(\alpha-1)(\alpha-2)}\right) = \frac{\alpha-1}{8(\alpha-2)}c_{\alpha-2}\end{aligned}$$

and for $\alpha = 1$ that

$$\begin{aligned}\varepsilon_1^9 &= -\frac{1}{4}(-1) + \frac{1}{2}\left(\frac{1}{2}\gamma\right) - \frac{1}{4}(\gamma+1) = 0, \\ \varepsilon_1^{10} &= -\frac{1}{8}(-1) - \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{4}\left(\frac{1}{2}\gamma\right) - \frac{1}{8}(\gamma+1) - \frac{1}{2}\left(-\frac{1}{2}\gamma - \frac{1}{2}\right) = 0.\end{aligned}$$

Assertions (1f) and (2f) follow from Lemmas 3.5, 3.2, and 3.6. \square

4. HEAT CONTENT ASYMPTOTICS FOR ROBIN BOUNDARY CONDITIONS

Section 4 is devoted to the proof of Assertion (3) of Theorem 1.5. Let $B = B_{\mathcal{R}}$ define Robin boundary conditions. We clear the previous notation concerning the constants ε_α^i . Recall $\tilde{B}_{\mathcal{R}}\rho = \rho_1 + \tilde{S}\rho_0$. Lemma 3.1 extends immediately to this setting, after including the additional tensor S in the Weyl calculus, to yield:

Lemma 4.1. *There exist universal constants ε_α^i and d_α^j so that:*

- (1) $\int_{\partial M} \beta_{0,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}})dy = \int_{\partial M} \varepsilon_\alpha^0 \langle \phi_0, \rho_0 \rangle dy.$
- (2) $\int_{\partial M} \beta_{1,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}})dy = \int_{\partial M} \{\varepsilon_\alpha^1 \langle \phi_1, \rho_0 \rangle + \varepsilon_\alpha^2 \langle L_{aa}\phi_0, \rho_0 \rangle + \varepsilon_\alpha^3 \langle \phi_0, \rho_1 \rangle + d_\alpha^1 \langle \phi_0, \tilde{B}_{\mathcal{R}}\rho \rangle\} dy.$
- (3) $\int_{\partial M} \beta_{2,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}})dy = \int_{\partial M} \{\varepsilon_\alpha^4 \langle \phi_2, \rho_0 \rangle + \varepsilon_\alpha^5 \langle L_{aa}\phi_1, \rho_0 \rangle + \varepsilon_\alpha^6 \langle E\phi_0, \rho_0 \rangle + \varepsilon_\alpha^7 \langle \phi_0, \rho_2 \rangle + \varepsilon_\alpha^8 \langle L_{aa}\phi_0, \rho_1 \rangle + \varepsilon_\alpha^9 \langle \text{Ric}_{mm}\phi_0, \rho_0 \rangle + \varepsilon_\alpha^{10} \langle L_{aa}L_{bb}\phi_0, \rho_0 \rangle + \varepsilon_\alpha^{11} \langle L_{ab}L_{ab}\phi_0, \rho_0 \rangle + \varepsilon_\alpha^{12} \langle \phi_{0;a}, \rho_{0;a} \rangle + \varepsilon_\alpha^{13} \langle \tau\phi_0, \rho_0 \rangle + \varepsilon_\alpha^{14} \langle \phi_1, \rho_1 \rangle + \langle d_\alpha^2 \phi_1 + d_\alpha^3 S\phi_0 + d_\alpha^4 L_{aa}\phi_0, \tilde{B}_{\mathcal{R}}\rho \rangle\} dy.$

We begin our analysis by showing that all the constants ε_α^i vanish:

Lemma 4.2.

- (1) *If $\tilde{B}_{\mathcal{R}}\rho = 0$, then $\partial_t \beta(\phi, \rho, D, B_{\mathcal{R}})(t) = -\beta(\phi, \tilde{D}\rho, D, B_{\mathcal{R}})(t)$.*
- (2) $\varepsilon_\alpha^0 = \varepsilon_\alpha^1 = \varepsilon_\alpha^2 = 0.$
- (3) $\varepsilon_\alpha^4 = \varepsilon_\alpha^5 = \varepsilon_\alpha^6 = \varepsilon_\alpha^7 = \varepsilon_\alpha^9 = \varepsilon_\alpha^{10} = \varepsilon_\alpha^{11} = \varepsilon_\alpha^{12} = \varepsilon_\alpha^{13} = 0.$
- (4) $\varepsilon_\alpha^3 = \varepsilon_\alpha^8 = \varepsilon_\alpha^{14} = 0.$

Proof. Assertion (1) follows using the same arguments used to prove Lemma 3.6 (1); Equation (3.h) then generalizes to become

$$\frac{1+k-\alpha}{2} \int_{\partial M} \beta_{k,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}})dy = - \int_{\partial M} \beta_{k-2,\alpha}^{\partial M}(\phi, \tilde{D}\rho, D, B_{\mathcal{R}})dy.$$

We take $S = 0$ and $\rho_1 = 0$; ρ_0 and ρ_2 are then arbitrary. Since $\beta_{-2,\alpha}^{\partial M} = 0$ and $\beta_{-1,\alpha}^{\partial M} = 0$, Assertion (2) follows. This implies that $\beta_{0,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}}) = 0$ and a similar argument now establishes Assertion (3). We now take $S = -1$ and $\rho_0 = \rho_1 = 1$ to establish Assertion (4). \square

The constants d_α^1 , d_α^2 , and d_α^3 can be determined by a 1-dimensional calculation. We adopt the following notational conventions. Let $M := [0, 1]$, let $A := \partial_x + b$ where $b \in C^\infty(M)$ is real valued, let $A^* := -\partial_x + b$, let $D_1 := A^*A$, let $D_2 := AA^*$, and let $B_{\mathcal{R}}\phi := A\phi|_{\partial M}$. The inward unit normal is ∂_x near $x = 0$ and $-\partial_x$ near $x = 1$. Thus this is a Robin boundary condition with $S(0) = b(0)$ and $S(1) = -b(1)$.

Lemma 4.3. *Let $\alpha \in \mathbb{C} - \mathbb{Z}$ with $\text{Re}(\alpha) < 0$. Adopt the notation established above.*

- (1) $\partial_t \beta(\phi, \rho, D_1, B_{\mathcal{R}})(t) = -\beta(A\phi, A\rho, D_2, B_{\mathcal{D}})(t).$
- (2) $\int_{\partial M} \beta_{k,\alpha}^{\partial M}(\phi, \rho, D_1, B_{\mathcal{R}}) dy = -\frac{2}{1+k-\alpha} \int_{\partial M} \beta_{k-1,\alpha+1}^{\partial M}(A\phi, A\rho, D_2, B_{\mathcal{D}}) dy.$
- (3) $d_\alpha^1 = \frac{2\alpha}{2-\alpha} c_{\alpha+1}.$
- (4) $d_\alpha^2 \phi_1 + d_\alpha^3 S\phi_0 = -\frac{2}{3-\alpha} c_\alpha \{(1-\alpha)\phi_1 + S\phi_0\}.$

Proof. We generalize the proof of Lemma 2.1.15 [9] where a similar result is established for $\alpha = 0$. One has that $Ae^{-tD_1, B_{\mathcal{R}}} = e^{-tD_2, B_{\mathcal{D}}} A$ on sufficiently smooth functions. Thus we may establish Assertion (1) by noting:

$$\begin{aligned} \partial_t \langle e^{-tD_1, B_{\mathcal{R}}} \phi, \rho \rangle &= -\langle A^* A e^{-tD_1, B_{\mathcal{R}}} \phi, \rho \rangle = -\langle A^* e^{-tD_2, B_{\mathcal{D}}} A \phi, \rho \rangle \\ &= -\langle e^{-tD_2, B_{\mathcal{D}}} A \phi, A \rho \rangle \end{aligned}$$

where the middle equality is justified by the boundary condition $B_{\mathcal{D}}$.

Since $r^\alpha \phi \in C^\infty(V)$, we have $r^{\alpha+1} A\phi \in C^\infty(V)$. Thus

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1+k-\alpha}{2} t^{(1+k-\alpha)/2-1} \int_{\partial M} \beta_{k,\alpha}^{\partial M}(\phi, \rho, D, B_{\mathcal{R}}) dy \\ &\sim -\sum_{\ell=0}^{\infty} t^{(1+\ell-(\alpha+1))/2} \int_{\partial M} \beta_{\ell,\alpha+1}^{\partial M}(A\phi, A\rho, D, B_{\mathcal{R}}) dy. \end{aligned}$$

Setting $\ell = k - 1$ and equating coefficients of $t^{(-1-k-\alpha)/2}$ yields Assertion (2).

The operators

$$D_1 = -(\partial_x^2 + (b' - b^2)) \quad \text{and} \quad D_2 = -(\partial_x^2 + (-b' - b^2))$$

determine flat connections. We suppose that b , ϕ , and ρ vanish identically near $r = 1$ so only the point $r = 0$ is relevant. We set $\phi_{-1} := 0$ and expand:

$$\phi \sim \sum_{i=0}^{\infty} \phi_i r^{i-\alpha}, \quad A\phi \sim \sum_{i=0}^{\infty} \{(i-\alpha)\phi_i + b\phi_{i-1}\} r^{i-\alpha-1}.$$

It now follows that $(A\phi)_0 = -\alpha\phi_0$ and $(A\phi)_1 = (1-\alpha)\phi_1 + b\phi_0$. We apply Assertion (2) with $k = 1$ and $k = 2$ and we apply Theorem 1.5 (1) to see:

$$\begin{aligned} \int_{\partial M} \langle d_\alpha^1 \phi_0, A\rho \rangle dy &= -\frac{2}{2-\alpha} c_{\alpha+1} \int_{\partial M} \langle -\alpha\phi_0, A\rho \rangle dy, \\ \int_{\partial M} \langle d_\alpha^2 \phi_1 + d_\alpha^3 b\phi_0, A\rho \rangle dy &= -\frac{2}{3-\alpha} c_\alpha \int_{\partial M} \langle (1-\alpha)\phi_1 + b\phi_0, A\rho \rangle dy. \end{aligned}$$

Assertions (2) and (3) now follow. \square

We extend Lemma 3.5 to the setting at hand to complete the proof of Theorem 1.5 (3).

Lemma 4.4. *Adopt the notation established in Lemma 3.5. Let $S = \frac{1}{2} \sum_a f'_a$ define Robin boundary conditions. Then*

- (1) $\int_{\partial M} \beta_{2,\alpha}^{\partial M}(\phi, \rho_M, D_M, B_{\mathcal{R}}) dy = 0.$
- (2) $d_\alpha^4 = -\frac{\alpha}{3-\alpha} c_\alpha.$

Proof. Taking into account the change in the connection, we have that $B_{\mathcal{R}}$ on M agrees with the pure Neumann operator $B_{\mathcal{N}}$ defined by $S = 0$ on $[0, 1]$. Since one has that $\rho dx = r\chi(r) dy dr$,

$$\beta(\phi, \rho, D_M, B_{\mathcal{R}})(t) = (2\pi)^{m-1} \beta(\phi, r\chi, -\partial_r^2, B_{\mathcal{N}})(t).$$

Assertion (1) now follows as $\int_{\partial[0,1]} \beta_{2,\alpha}^{\partial M}(\phi, r\chi, -\partial_r^2, B_{\mathcal{N}}) dr = 0$.

To prove Assertion (2), it is simply a matter of disentangling everything. We use the equations of structure derived in the proof of Lemma 3.4 to see:

$$\begin{aligned} \phi_0 &= 1, & \phi_1 &= -\frac{1}{2} \sum_a f'_a, & \rho_0 &= 0, & \rho_1 &= 1, \\ S &= \frac{1}{2} \sum_a f'_a, & L_{aa} &= -\sum_a f'_a. \end{aligned}$$

We may now compute:

$$0 = \int_{\partial M} \left\{ -\frac{2}{3-\alpha} c_\alpha \left\{ (1-\alpha) \left(-\frac{1}{2} \sum_a f'_a \right) + \frac{1}{2} \sum_a f'_a \right\} + d_\alpha^4 \left(-\sum_a f'_a \right) \right\} dy.$$

It now follows that $d_\alpha^4 = -\frac{\alpha}{3-\alpha} c_\alpha$. \square

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